

TOPOLOGICAL RESOLUTIONS IN $K(2)$ -LOCAL HOMOTOPY THEORY AT THE PRIME 2

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ABSTRACT. We provide a topological duality resolution for the spectrum $E_2^{h\mathbb{S}_2^1}$, which itself can be used to build the $K(2)$ -local sphere. The resolution is built from spectra of the form E_2^{hF} where E_2 is the Morava spectrum for the formal group of a supersingular curve at the prime 2 and F is a finite subgroup of the automorphisms of that formal group. The results are in complete analogy with the resolutions of [GHR05] at the prime 3, but the methods are of necessity very different. As in the prime 3 case, the main difficulty is in identifying the top fiber; to do this, we make calculations using Henn's centralizer resolution.

Chromatic stable homotopy theory uses the algebraic geometry of smooth one-parameter formal groups to organize calculations and the search for large scale phenomena. In particular, the chromatic filtration on the category of the p -local finite spectra corresponds to the height filtration for formal group laws. The layers of the chromatic filtration are given by localization with respect to the Morava K -theories $K(n)$, with $n \geq 0$. Thus, to understand the homotopy type of a finite spectrum X we begin by addressing $L_{K(n)}X$ for all prime numbers p and all $0 \leq n < \infty$. A useful and inspirational guide to this point of view can be found in the table in section 2 of [HG94].

If $n = 0$, $K(0) = H\mathbb{Q}$ and L_0X is the rational homotopy type of X . For $n \geq 1$, the basic computational tool in $K(n)$ -local homotopy theory is the $K(n)$ -local Adams-Novikov Spectral Sequence

$$H^s(\mathbb{G}_n, (E_n)_t X) \implies \pi_{t-s} L_{K(n)} X.$$

Here \mathbb{G}_n is the automorphism group of a pair (\mathbb{F}_q, Γ_n) where \mathbb{F}_q is a finite field of characteristic p and Γ_n is a chosen formal group of height n over \mathbb{F}_q . Then E_n is the Morava (or Lubin-Tate) E -theory defined by (\mathbb{F}_q, Γ_n) . We will give more details and make precise choices in section 1.

First suppose p is large with respect to n . (To be precise, we may take $2p - 2 \geq \max\{n^2, 2n + 2\}$ or $p > 3$ if $n=2$.) Then the Adams-Novikov Spectral Sequence for $X = S^0$ collapses and will have no extensions, so the problems becomes algebraic, although by no means easy. See for example, [SY95], [Beh12], or [Lad13], for the case $n = 2$ and $p > 3$. However, if p is small with respect to n , the group \mathbb{G}_n has finite subgroups of p -power order and the spectral sequence will usually have differentials and extensions, so the problem is no longer purely algebraic. At this point, topological resolutions become a useful way to organize the contributions of

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the finite subgroups. The key to unlocking this idea is the Hopkins-Miller theorem, which implies that \mathbb{G}_n , and hence all of its finite subgroups, act on E_n and that $L_{K(n)}S^0 \simeq E_n^{h\mathbb{G}_n}$. See [DH04] for this and more.

The prototypical example is at $n = 1$ and $p = 2$. Adams and Baird [Bou79], and Ravenel [Rav84] showed that here we have a fiber sequence

$$L_{K(1)}S^0 \rightarrow KO \xrightarrow{\psi^3-1} KO$$

where KO is 2-complete real K -theory. For a suitable choice of Γ_1 we can take $E_1 = K$, where K is 2-complete complex K -theory, $\mathbb{G}_1 = \mathbb{Z}_2^\times$ is the units in the 2-adic integers, and $C_2 = \{\pm 1\} \subseteq \mathbb{Z}_2^\times$ acts through complex conjugation. Then we can rewrite this fiber sequence as

$$L_{K(1)}S^0 = E_1^{h\mathbb{G}_1} \longrightarrow E_1^{hC_2} \xrightarrow{\psi^3-1} E_1^{hC_2},$$

and ψ^3 is a topological generator of $\mathbb{Z}_2^\times / \{\pm 1\} \simeq \mathbb{Z}_2$.

For higher heights the topological resolutions will not be simple fiber sequences, but finite towers of fibrations with the successive fibers built from E^{hF_i} where F_i runs over various finite subgroups of \mathbb{G}_n .

In [GHMR05] the authors generalized the fiber sequence of the $K(1)$ -local case to the case $n = 2$ and $p = 3$. One way to say what they proved is the following. Let us write $E = E_2$ to simplify notation. First we have a split short exact sequence of groups

$$\{1\} \rightarrow \mathbb{G}_2^1 \longrightarrow \mathbb{G}_2 \xrightarrow{N} \mathbb{Z}_3 \rightarrow \{1\}$$

where N is obtained from a determinant (See (1.6)) and hence a fiber sequence

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\psi-1} E^{h\mathbb{G}_2^1},$$

where ψ is any element of \mathbb{G}_2 which maps by N to a generator of \mathbb{Z}_3 . Then, second, there exists a resolution of $E^{h\mathbb{G}_2^1}$

$$E^{h\mathbb{G}_2^1} \rightarrow E^{hG_{24}} \rightarrow \Sigma^8 E^{hSD_{16}} \rightarrow \Sigma^{40} E^{hSD_{16}} \rightarrow \Sigma^{48} E^{hG_{24}}$$

Here *resolution* means each successive composition is null-homotopic and all possible Toda brackets are zero modulo indeterminacy; thus, the sequence refines to a tower of fibrations

$$\begin{array}{ccccccc} E^{h\mathbb{G}_2^1} & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & E^{hG_{24}} \\ \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{45} E^{hG_{24}} & & \Sigma^{38} E^{hSD_{16}} & & \Sigma^7 E^{hSD_{16}} & & \end{array}$$

The maximal finite subgroup of \mathbb{G}_2 of 3-power order is a cyclic group C_3 of order 3; it is unique up to conjugation in \mathbb{G}_2^1 . The group G_{24} is the maximal finite subgroup of \mathbb{G}_2 containing C_3 . The subgroup SD_{16} is the semidihedral group of order 16. Because of the symmetry of this resolution, and because \mathbb{G}_2^1 is a virtual Poincaré duality group of dimension 3, this is called a *duality resolution*. This resolution and related resolutions were instrumental in exploring the $K(2)$ -local category at primes $p > 2$. See [GHM04], [HKM13], [Beh06], [GHM14], [GHMR15],

[GH16], and [Lad13]. The latter paper makes a thorough exploration of what happens at $p > 3$.

The main theorem of this paper provides an analog of the duality resolution at $n = 2$ and $p = 2$. The prime 2 is much harder for a number of reasons. First, the maximal finite 2-subgroup of \mathbb{G}_2 is not cyclic, but isomorphic to Q_8 , the quaternion group of order 8. Second, every finite subgroup of \mathbb{G}_2 that we consider contains the central $C_2 = \{\pm 1\}$ of \mathbb{G}_2 ; therefore, the homotopy groups of the relevant fixed point spectra E^{hF} are much more complicated. This means that the strategy of proof of [GHMR05] won't work and we need to find another way.

We now state our main result. As our chosen formal group Γ_2 we will use the formal group of a supersingular curve over \mathbb{F}_4 . The curve will be defined over \mathbb{F}_2 , so that $\mathbb{G}_2 \cong \mathbb{S}_2 \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ where $\mathbb{S}_2 = \text{Aut}(\Gamma_2/\mathbb{F}_4)$ is the group of automorphisms of Γ_2 over \mathbb{F}_4 . Once again we have fiber sequence

$$L_{K(2)}S^0 \longrightarrow E^{h\mathbb{G}_2^1} \xrightarrow{\psi^{-1}} E^{h\mathbb{G}_2^1}.$$

We have $E^{h\mathbb{G}_2^1} = (E^{h\mathbb{S}_2^1})^{h\text{Gal}}$ where $\text{Gal} = \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ and $\mathbb{S}_2^1 = \mathbb{S}_2 \cap \mathbb{G}_2^1$. For many computational applications, the difference between $E^{h\mathbb{G}_2^1}$ and $E^{h\mathbb{S}_2^1}$ is innocuous. See Lemma 1.28. Then our main result is this:

Theorem 0.1. *There exists a resolution of $E^{h\mathbb{S}_2^1}$ in the $K(2)$ -local category at the prime 2*

$$E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow \Sigma^{48} E^{hC_6} \rightarrow \Sigma^{48} E^{hG_{24}}.$$

The spectrum E^{hC_6} is 48-periodic, so $E^{hC_6} \simeq \Sigma^{48} E^{hC_6}$; this susension is there only to emphasize the symmetry in the resolution.

Once again resolution means each successive composition is null-homotopic and all possible Toda brackets are zero modulo indeterminacy; thus, the sequence again refines to a tower of fibrations. The group $G_{24} \subseteq \mathbb{S}_2^1$ is a maximal finite subgroup of \mathbb{S}_2^1 containing Q_8 ; it is not unique, even up to conjugation, and in fact the two different conjugacy classes naturally appear in the related algebraic resolution [Bea15a]. See Theorem 3.15. The maximal finite subgroup $G_{48} = G_{24} \rtimes \text{Gal}$ of \mathbb{G}_2 containing Q_8 is of order 48 and is unique up to conjugation; hence, the homotopy type of $E^{hG_{24}}$ is well-defined.

This result has already had applications: it is an ingredient in Agnès Beaudry's analysis of the Chromatic Splitting Conjecture at $p = n = 2$. See [Bea15b].

The apparent similarity of Theorem 0.1 with the prime 3 analog is tantalizing, especially the suspension factor on the last term, but we as yet have no conceptual explanation. The proof at the prime 2 can be adapted to the prime 3 but in both cases it comes down to a very specific, prime dependent calculation.

A very satisfying feature of chromatic height 2 is the connection with the theory of elliptic curves. The subgroup $G_{24} \subseteq \mathbb{S}_2$ is the automorphism group of our chosen supersingular curve inside the automorphisms of its formal group. Appealing to Strickland [Str] we can use this to get formulas for the action of G_{24} on E_* , a

necessary beginning to group cohomology calculations. Furthermore, we know that

$$E^{hG_{48}} \simeq (E^{hG_{24}})^{h\text{Gal}} \simeq L_{K(2)}Tmf$$

where Tmf is the global sections of the sheaf of E_∞ -ring spectra on the compactified stack of generalized elliptic curves provided by Hopkins and Miller. See [DFHH14]. Similarly, E^{hC_6} is the localization of global sections of the similar sheaf for elliptic curves with a level 3 structure. See [MR09]. We won't need anything like the full power of the Hopkins-Miller theory here, although we do use some of the calculations that arise from this point of view. See Section 2.

We will prove Theorem 0.1 in three steps.

First we prove, in Section 3, that there exists a resolution

$$E^{hS_2^1} \rightarrow E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow E^{hC_6} \rightarrow X$$

where

$$E_*X \cong E_*E^{hG_{24}}$$

as twisted \mathbb{G}_2 -modules. This resolution and the necessary algebraic preliminaries were announced in [Hen07], and grew out of the work surrounding [GHMR05]. The algebraic preliminaries are discussed in detail in [Bea15a].

Second, in section 4 we examine the Adams-Novikov Spectral Sequence

$$H^*(G_{24}, E_*) \implies \pi_*X.$$

Using a comparison of our resolution with a second resolution, due to Henn, we show, roughly, that certain classes $\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$ are permanent cycles—which would certainly be necessary if our main result is true. The exact result is in Corollary 4.7. These calculations were among the main results in the first author's thesis [Bob14] and the key ideas for the entire project can be found there. This ratifies a comment of Mark Mahowald that there is a class in $\pi_{44}L_{K(2)}S^0$ which supports non-zero multiplications by η and $\bar{\kappa}$ and the only way this could happen is if Δ^2 is a permanent cycle. This insight was, we think, the result of long hours of contemplation of the results of Shimomura and Wang [SW02] and, indeed, one reason for this entire project is to find some way to catch up with those amazing calculations.

Third and finally, in section 5 we use a variation of this same comparison argument to produce the equivalence $\Sigma^{48}E^{hG_{24}} \simeq X$. See Theorem 5.8. At the very end we add a remark about the possibility or impossibility of resolutions for $L_{K(2)}S^0$ itself. See Remark 5.10.

There is a number of sections of preliminaries. Section 1 provides the usual background on the $K(n)$ -local category as well as some more specific information on mapping spaces. We also prove that if $K \subseteq \mathbb{S}_n = \text{Aut}(\Gamma_n/\mathbb{F}_{p^n})$ is any closed subgroup closed under the action of the Galois group $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, then there is a Gal-equivariant equivalence

$$\text{Gal}_+ \wedge E_n^{h(K \rtimes \text{Gal})} \simeq E_n^{hK}.$$

See Lemma 1.28. This fact seems to be well-known, but it hard to find explicitly in print. But see (0.3) of [DH95]. Section 2 pulls together what we need about the

homotopy groups of various fixed point spectra. This draws from many sources, and we try to be complete there.

An essential ingredient in our argument is the existence of the other topological resolution for $E^{h\mathbb{S}_2^1}$, Henn’s centralizer resolution from §3.4 of [Hen07]. We have less control over the maps in this resolution, but, as has happened before ([GHM04]; see also [GH16]), this resolution provides essential information needed to solve ambiguities in the duality resolution. It is also much closer to being an Adams-Novikov-style resolution as it is based on relative homological algebra. We cover some of this material at the end of section 3.

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1. RECOLLECTIONS ON THE $K(n)$ -LOCAL CATEGORY

We begin with the standard material on the $K(n)$ -local category, the Morava stabilizer group, and Morava E -theory, also known as Lubin-Tate theory. We then get specific at $n = 2$ and $p = 2$, discussing the role of formal groups arising from supersingular elliptic curves. We add some material on the homotopy type of the spectrum of maps between various fixed point spectra derived from Morava E -theory and then, finally, we discuss, the role of the Galois group.

1.1. The $K(n)$ -local category. Fix a prime p and let $n \geq 1$. Let Γ_n be a formal group of height n over the finite field \mathbb{F}_p of p elements. Then for any finite extension $i : \mathbb{F}_p \subseteq \mathbb{F}_q$ of \mathbb{F}_p , we form the group $\text{Aut}(\Gamma_n/\mathbb{F}_q)$ of the automorphisms of $i^*\Gamma_n$

over \mathbb{F}_q . We fix a choice of Γ_n with the property that any extension $\mathbb{F}_{p^n} \subseteq \mathbb{F}_q$ gives an isomorphism

$$\mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \xrightarrow{\cong} \mathrm{Aut}(\Gamma_n/\mathbb{F}_q).$$

The usual Honda formal group satisfies these criteria: this has a formal group law which is p -typical and with p -series $[p](x) = x^{p^n}$. However, if $n = 2$, then the formal group of a supersingular elliptic curve defined over \mathbb{F}_p will also do, and this will be our preferred choice at $p = 2$. Define

$$(1.1) \quad \mathbb{S}_n \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\Gamma_n/\mathbb{F}_{p^n}).$$

If we choose a coordinate for Γ_n , then any element \mathbb{S}_n defines power series $\phi(x) \in x\mathbb{F}_{p^n}[[x]]$ invertible under composition, and assignment $\phi(x) \mapsto \phi'(0)$ defines a map

$$\mathbb{S}_n \longrightarrow \mathbb{F}_{p^n}^\times.$$

For the Honda formal group this is a split surjection; at $n = p = 2$ we will choose a supersingular curve so that this is also true for the associated formal group. See (1.9) below. In any case, we assume this for Γ_n . Then we define S_n to be the kernel of this map; this is the p -Sylow subgroup of the profinite group \mathbb{S}_n . Our assumptions give an isomorphism $S_n \rtimes \mathbb{F}_{p^n}^\times \cong \mathbb{S}_n$.

Define the (big) *Morava stabilizer group* \mathbb{G}_n as the automorphism group of the pair $(\mathbb{F}_{p^n}, \Gamma_n)$. Since Γ_n is defined over \mathbb{F}_p , there is an isomorphism

$$(1.2) \quad \mathbb{G}_n \cong \mathrm{Aut}(\Gamma/\mathbb{F}_{p^n}) \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) = \mathbb{S}_n \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p).$$

We will often write $\mathrm{Gal} = \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ when the field extension is understood.

We next must define Morava K -theory; there are many variants, all of which have the same Bousfield class and define the same localization; we will choose a variant which works well with Morava E -theory. Let $K(n) = K(\mathbb{F}_{p^n}, \Gamma_n)$ be the 2-periodic ring spectrum with homotopy groups

$$K(n)_* = \mathbb{F}_{p^n}[u^{\pm 1}]$$

and with associated formal group Γ_n . Here the class u is in degree -2 . The group $F_0 = \mathbb{F}_{p^n}^\times \rtimes \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on $K(n)$ and

$$\mathbb{F}_p[v_n^{\pm 1}] \cong (K(n)^{hF_0})_* = K(n)_*^{F_0}$$

where $v_n = u^{-(p^n-1)}$. The spectrum $K(n)^{hF_0}$ is thus a more classical version of Morava K -theory.

We will spend a great deal of time working in the $K(n)$ -local category and, when doing so, all our spectra will implicitly be localized. In particular, we emphasize that we often write $X \wedge Y$ for $L_{K(n)}(X \wedge Y)$, as this is the smash product internal to the $K(n)$ -local category.

We now define the Morava spectrum $E = E_n = E(\mathbb{F}_{p^n}, \Gamma_n)$. This is the complex oriented, Landweber exact, 2-periodic, E_∞ -ring spectrum with

$$(1.3) \quad E_* = (E_n)_* \cong \mathbb{W}[[u_1, \dots, u_{n-1}]] [u^{\pm 1}]$$

with u_i in degree 0 and u in degree -2 . Here $\mathbb{W} = W(\mathbb{F}_{p^n})$ is the Witt vectors on \mathbb{F}_{p^n} . Note that E_0 is a complete local ring with residue field \mathbb{F}_{p^n} ; the formal group over E_0 is a choice of universal deformation of the formal group Γ_n over \mathbb{F}_{p^n} . (We

will be specific about this choice at $n = p = 2$ below in subsection 1.2.) The group $\mathbb{G}_n = \text{Aut}(\Gamma_n) \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ acts on E , by the Hopkins-Miller theorem [GH04] and we have, by [DH04], a spectral sequence for any closed subgroup $F \subseteq \mathbb{G}_n$,

$$(1.4) \quad H^s(F, E_t) \implies \pi_{t-s} E^{hF}.$$

We will collectively call these by the name Adams-Novikov Spectral Sequence. If $F = \mathbb{G}_n$ itself, then $E^{h\mathbb{G}_n} \simeq L_{K(n)} S^0$ and we are computing the homotopy groups of the $K(n)$ -local sphere.

Since \mathbb{G}_n acts on E , \mathbb{G}_n acts on

$$E_* X \stackrel{\text{def}}{=} \pi_* L_{K(n)}(E \wedge X).$$

The E_* -module $E_* X$ is equipped with the \mathfrak{m} -adic topology where \mathfrak{m} is the maximal ideal in E_0 . This topology is always topologically complete, but need not be separated. With respect to this topology, the group \mathbb{G}_n acts through continuous maps and the action is twisted because it is compatible with the action of \mathbb{G}_n on the coefficient ring E_* . See [GHMR05] §2 for some precise assumptions which guarantee that $E_* X$ is complete and separated. All modules which will be used in this paper will in fact satisfy these assumptions, and we will call these modules *twisted \mathbb{G}_n -modules*, or *Morava modules*.

For example, let $F \subseteq \mathbb{G}_n$ be a closed subgroup. Then, by many references (see Theorem 12 of [Str00], for example) there is an isomorphism of twisted \mathbb{G}_n -modules

$$E_* E^{hF} \cong \text{map}(\mathbb{G}_n/F, E_*)$$

where $\text{map}(-, -)$ denotes the set of continuous maps. On the right hand side of this equation, E_* acts on the target and the \mathbb{G}_n -action is diagonal. See Remark 1.17 for more on this subject.

Various subgroups of \mathbb{G}_n will play a role in this paper, especially at $n = 2$ and $p = 2$. The right action of $\text{Aut}(\Gamma_n)$ on $\text{End}(\Gamma_n)$ defines a determinant map $\det : \mathbb{S}_n = \text{Aut}(\Gamma_n/\mathbb{F}_{p^n}) \rightarrow \mathbb{Z}_p^\times$ which extends to a determinant map

$$(1.5) \quad \mathbb{G}_n \cong \mathbb{S}_n \rtimes \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{\det \times 1} \mathbb{Z}_p^\times \times \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \xrightarrow{p_1} \mathbb{Z}_p^\times.$$

Define the *reduced determinant* (or *reduced norm*) N to be the composition

$$(1.6) \quad \begin{array}{ccc} \mathbb{G}_n & \xrightarrow{\det} & \mathbb{Z}_p^\times \longrightarrow \mathbb{Z}_p^\times / C \cong \mathbb{Z}_p. \\ & \searrow N & \uparrow \\ & & \mathbb{Z}_p^\times \end{array}$$

where $C \subseteq \mathbb{Z}_p^\times$ is the finite subgroup. For example, $C = \{\pm 1\}$ if $p = 2$. There are isomorphisms $\mathbb{Z}_p^\times / C \cong \mathbb{Z}_p$ and we choose one. Write \mathbb{G}_n^1 for the kernel of N , $\mathbb{S}_n^1 = \mathbb{S}_n \cap \mathbb{G}_n^1$, and $S_n^1 = S_n \cap \mathbb{G}_n^1$. The map $N : S_n \rightarrow \mathbb{Z}_p$ is split surjective and we have semi-direct product decompositions for each of the groups \mathbb{G}_n , \mathbb{S}_n , and S_n ; for example, there is an isomorphism

$$\mathbb{S}_n^1 \rtimes \mathbb{Z}_p \cong \mathbb{S}_n.$$

If n is prime to p , we can choose a central splitting and the semi-direct product is actually a product, but that is not the case of interest here.

1.2. Deformations from elliptic curves. Here we spell out what we need from the theory of elliptic curves at $p = 2$; this will give us a preferred formal group and a preferred universal deformation. Choose Γ_2 to the formal group obtained from the elliptic curve C_0 over \mathbb{F}_2 defined by the Weierstrass equation

$$(1.7) \quad y^2 + y = x^3.$$

This is a standard representative for the unique isomorphism class of supersingular curves over $\overline{\mathbb{F}}_2$; see [Sil09], Appendix A. Because C_0 is superingular, Γ_2 has height 2, as the notation indicates. Following [Str] let C be the elliptic curve over $\mathbb{W}(\mathbb{F}_4)[[u_1]]$ defined by the Weierstrass equation

$$(1.8) \quad y^2 + 3u_1xy + (u_1^3 - 1)y = x^3.$$

This reduces to C_0 modulo the maximal ideal $\mathfrak{m} = (2, u_1)$; the formal group G of C is a choice of the universal deformation of Γ_2 . Using the formulas of [Sil09], §III.1, we can calculate that we have an equation for the modular form c_4 for C

$$c_4(C) = 9u_1(u_1^3 + 8)u^{-4}.$$

This implies that $v_1 = v_1(G) \equiv u_1u^{-1} \in E_2/2$, as $v_1^4 \equiv c_4$.

Again turning to [Sil09], Appendix A, and specifically to Proposition 1.2 there, we have

$$(1.9) \quad G_{24} \stackrel{\text{def}}{=} \text{Aut}(C_0/\mathbb{F}_4) \cong Q_8 \rtimes \mathbb{F}_4^\times$$

where $\mathbb{F}_4^\times \cong C_3$ acts on Q_8 as the 3-Sylow subgroup of $\text{Aut}(Q_8)$. Define

$$(1.10) \quad G_{48} \stackrel{\text{def}}{=} \text{Aut}(\mathbb{F}_4, C_0) \cong \text{Aut}(C_0/\mathbb{F}_4) \rtimes \text{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

Since any automorphism of the pair (\mathbb{F}_4, C_0) induces an automorphism of the pair (\mathbb{F}_4, Γ_2) we get a map $G_{48} \rightarrow \mathbb{G}_2$. This map is an injection and we identify G_{48} with its image. We will be interested in various subgroups of G_{48} .

Remark 1.11. The following subgroups will play an important role in this paper.

- (1) $C_2 = \{\pm 1\} \subseteq Q_8$;
- (2) $C_6 = C_2 \times \mathbb{F}_3^\times$;
- (3) C_4 , any of the subgroups of order 4 in Q_8 ;
- (4) G_{24} and G_{48} themselves.

The subgroup C_4 is not unique, but it is unique up to conjugation in G_{24} , so that the homotopy type of E^{hC_4} is well-defined.

Remark 1.12. We have been discussing G_{24} as a subgroup of \mathbb{S}_2 , but it can also be thought of as a quotient as well. Inside of \mathbb{S}_2 there is a normal torsion-free pro-2-subgroup K which has the property that the composition

$$G_{24} \longrightarrow \mathbb{S}_2 \longrightarrow \mathbb{S}_2/K$$

is an isomorphism. Thus we have a decomposition $K \rtimes G_{24} \cong \mathbb{S}_2$. See [Bea15a] for details. The group K is a Poincaré duality group of dimension 4.

Remark 1.13. There is a map of Adams-Novikov Spectral Sequences

$$\begin{array}{ccc} \mathrm{Ext}_{BP_*BP}^s(\Sigma^t BP_*, BP_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s} S^0 \\ \downarrow & & \downarrow \\ H^s(\mathbb{G}_2, E_t) & \Longrightarrow & \pi_{t-s} L_{K(2)} S^0 \end{array}$$

but it takes a little care to define. Let $G(x, y) \in E^0[[x, y]]$ be the formal group law of the supersingular curve of (1.8); since G is the formal group of a Weierstrass curve, it has a preferred coordinate. Let $G_*(x, y) = uG(u^{-1}x, u^{-1}y) \in E^*[[x, y]]$. Then G_* is a formal group over E^* with coordinate in cohomological degree 2 and, therefore, is classified by a map $\mathbb{Z}_{(2)} \otimes MU_* \rightarrow E_*$. Since G_* is not evidently 2-typical, it need not be classified by a map $BP_* \rightarrow E_*$. However, over a $\mathbb{Z}_{(2)}$ -algebra, the Cartier idempotent gives an equivalence between the groupoid of all formal group laws and the groupoid of 2-typical formal group laws; hence we have a diagram of spectral sequences as needed:

$$\begin{array}{ccc} \mathrm{Ext}_{BP_*BP}^s(\Sigma^t BP_*, BP_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s} S^0 \\ \cong \uparrow & & \uparrow = \\ \mathbb{Z}_{(2)} \otimes \mathrm{Ext}_{MU_*MU}^s(\Sigma^t MU_*, MU_*) & \Longrightarrow & \mathbb{Z}_{(2)} \otimes \pi_{t-s} S^0 \\ \downarrow & & \downarrow \\ H^s(\mathbb{G}_2, E_t) & \Longrightarrow & \pi_{t-s} L_{K(2)} S^0. \end{array}$$

We will use this below in the section on the cohomology of G_{48} .

Remark 1.14. In the proof of Lemma 1.27 we will use that the spectral sequence

$$H^s(\mathbb{G}_n, E_t X) \Longrightarrow \pi_{t-s} L_{K(n)} X$$

has a horizontal vanishing line at E_∞ for X a finite spectrum or even a finite $K(n)$ -local spectrum. The standard reference for this fact is [DH04], Proposition A.3, although that result is not explicit about vanishing lines. What that result does say is that $L_{K(n)} S^0$ is in the smallest class \mathcal{C} of spectra containing E and closed under cofibrations and retracts. Arguing as in [HPS99] we can conclude that any spectrum in \mathcal{C} has a horizontal vanishing line.

Alternatively, we could use the techniques of [HPS99] to show that the collection of finite X for which the vanishing line appears is a thick subcategory. Then we need only show there exists a finite spectrum Z with $H_*(Z, \mathbb{Q}) \neq 0$ and so that $H^s(\mathbb{G}_n, E_* Z) = 0$ has a horizontal vanishing line. The existence of the needed spectra Z can be deduced from the construction of §8.3 of [Rav92]. At $n = 2$, where we work in this paper, we can be explicit. If $p \geq 5$, we can take $Z = S^0$, if $p = 3$, we can take Z to be a 3-cell 8-skeleton of BP , and if $p = 2$, we can take Z to be a double of $A(1)$. See [Mat13], although these constructions were known to experts such as Hopkins and Ravenel long before that paper.

We also note that [DH04], Proposition A.3 implies that the functor

$$X \mapsto E_* X = \pi_* L_{K(n)}(E_n \wedge X)$$

detects weak equivalences in the $K(n)$ -local category. Given a map $f : X \rightarrow Y$, the class of spectra Z so that $L_{K(n)}(Z \wedge f)$ is a weak equivalence is closed under cofibrations and retracts; hence if E_n is in this class, then $L_{K(n)}S^0$ is in this class.

1.3. Mapping spectra. We collect here some basics about the mapping spectra $F(E^{hF_1}, E^{hF_2})$ for various subgroups F_1 and F_2 of \mathbb{G}_n .

Remark 1.15. Let $F \subseteq \mathbb{G}_n$ be a closed subgroup. We begin with the equivalence

$$(1.16) \quad E \wedge E^{hF} \simeq \text{map}(\mathbb{G}/F, E)$$

from the local smash product to the localized spectrum of continuous maps. It is helpful to visualize this map as sending $x \wedge y$ to the function $gF \mapsto x(gy)$. We will continue this mnemonic below: using point-wise defined functions to indicate maps of spectra which cannot be defined that way. We hope the readers can fill in the details themselves; if not, complete details can be found in [GHMR05] §2.

The action of \mathbb{G}_n on the left factor of E in (1.16) defines the Morava module structure of E_*E^{hF} ; this corresponds to the diagonal action on the functions (the right factor):

$$(h\phi)(g) = h\phi(h^{-1}g).$$

Note that

$$(E_*E^{hF})^{\mathbb{G}_n} = \text{map}_{\mathbb{G}_n}(\mathbb{G}_n/F, E_*) \cong (E_*)^F.$$

This extends to an isomorphism

$$H^*(\mathbb{G}_n, E_*E^{hF}) \cong H^*(F, E_*).$$

Remark 1.17. We now recall some results from [GHMR05]. If $X = \lim X_i$ is a profinite set, let $E[[X]] = \lim E \wedge X_i^+$ where the $+$ indicates a disjoint basepoint. Then if F_1 is a closed subgroup of \mathbb{G}_n we have an equivalence

$$(1.18) \quad E[[\mathbb{G}_n/F_1]] \simeq F(E^{hF_1}, E)$$

defined as follows. Let F_E denote the function spectrum in E -modules. Then

$$\begin{aligned} E[[\mathbb{G}_n/F_1]] &\simeq F_E(\text{map}(\mathbb{G}_n/F_1, E), E) \\ &\simeq F_E(E \wedge E^{hF_1}, E) \\ &\simeq F(E^{hF_1}, E). \end{aligned}$$

Next note that the equivalence of (1.18) is \mathbb{G}_n -equivariant with the following actions: in $F(E^{hF_1}, E)$ we act on the target and in $E[[\mathbb{G}_n/F_1]]$ we act as follows:

$$h\left(\sum a_g g F_1\right) = \sum h(a_g) h^{-1} g F_1.$$

We can now make the following deductions. First suppose $F_1 = U$ is open (and hence closed), so that $\mathbb{G}_n/F_1 = \mathbb{G}_n/U$ is finite. Let F_2 be finite. Then we have equivalences

$$(1.19) \quad \begin{aligned} \prod_{F_2 \setminus \mathbb{G}_n/U} E^{hF_x} &\simeq E[[\mathbb{G}_n/U]]^{hF_2} \\ &\simeq F(E^{hU}, E)^{hF_2} \\ &\simeq F(E^{hU}, E^{hF_2}). \end{aligned}$$

The product in the source is over the double coset space, and for a double coset F_2xU ,

$$F_x = F_2 \cap xUx^{-1} \subseteq F_2.$$

Note that F_x is defined only up to conjugation, but the fixed point spectrum E^{hF_x} is well-defined up to weak equivalence. The first map of (1.19) sends $a \in E^{hF_x}$ to the sum

$$\sum_{gF_x \in F_2/F_x} (g^{-1}a) gxU.$$

We say a word about the naturality of the equivalence of (1.19). Suppose $U \subseteq V \subseteq \mathbb{G}_n$ is a nested pair of open subgroups. Then for each double coset F_2xU we get a double coset F_2xV , a nested pair of subgroups

$$F_x = F_2 \cap xUx^{-1} \subseteq F_2 \cap xVx^{-1} = G_x,$$

and a transfer map

$$\mathrm{tr}_x : E^{hF_x} \longrightarrow E^{hG_x}$$

associated to this inclusion. Then we have a commutative diagram

$$(1.20) \quad \begin{array}{ccc} \prod_{F_2 \backslash \mathbb{G}_n / U} E^{hF_x} & \xrightarrow{\simeq} & E[[\mathbb{G}_n / U]]^{hF_2} \\ \mathrm{tr} \downarrow & & \downarrow g \\ \prod_{F_2 \backslash \mathbb{G}_n / V} E^{hG_x} & \xrightarrow{\simeq} & E[[\mathbb{G}_n / V]]^{hF_2} \end{array}$$

where the map tr is the sum of the transfer maps and the map g induced by the quotient $\mathbb{G}_n / U \rightarrow \mathbb{G}_n / V$.

For a more general closed subgroup F_1 write $F_1 = \cap_i U_i$ where $U_i \subseteq \mathbb{G}_n$ is open. Then, for F_2 finite we get a weak equivalence

$$(1.21) \quad \mathrm{holim}_i \prod_{F_2 \backslash \mathbb{G}_n / U_i} E^{hF_x} \simeq F(E^{hF_1}, E^{hF_2})$$

where the product is as before and

$$F_x = F_2 \cap xU_i x^{-1} \subseteq F_2$$

depends on i and, as in (1.20), there may be transfer maps in the transition maps for the homotopy limit. As F_2 is finite and the product in (1.21) is finite, this implies that the image of

$$\prod_{F_2 \backslash \mathbb{G}_n / U_j} \pi_* E^{hF_x} \longrightarrow \prod_{F_2 \backslash \mathbb{G}_n / U_i} \pi_* E^{hF_x}$$

is independent of j for large j . It follows that there will be no \lim^1 term for the homotopy groups of the inverse limit and hence there is an isomorphism

$$(1.22) \quad \lim_i \prod_{F_2 \backslash \mathbb{G}_n / U_i} \pi_* E^{hF_x} \cong \pi_* F(E^{hF_1}, E^{hF_2})$$

Lemma 1.23. *Let $F_2 \subseteq \mathbb{G}_n$ be a finite subgroup and let $F_1 \subseteq \mathbb{G}_n$ be a closed subgroup. Let $N \subseteq \mathbb{G}_2$ be the subset*

$$N = \{x \in \mathbb{G}_n \mid x^{-1}F_2x \subseteq F_1\}.$$

Then F_1 acts on the right on N and for every $x \in N/F_1$ we have map

$$f_x : E^{hF_2} \rightarrow F(E^{hF_1}, E^{hF_2})$$

splitting the augmentation $\iota : F(E^{hF_1}, E^{hF_2}) \rightarrow E^{hF_2}$.

Note that $N/F_1 \cong \text{map}_{\mathbb{G}_n}(\mathbb{G}_n/F_2, \mathbb{G}_n/F_1)$. If X is any continuous \mathbb{G}_n -set and $aF_1 \in N/F_1$, then left multiplication by a determines a map $a : X^{F_1} \rightarrow X^{F_2}$.

Proof. The results of Lemma 1.23 can be read off the weak equivalence of (1.21), but it's simpler than that: left multiplication by a defines a class $g_a : S^0 \rightarrow F(E^{hF_1}, E^{hF_2})$ and the target is an E^{hF_2} module. \square

1.4. The action of the Galois group. We now turn to analyzing $E^{h\mathbb{S}_2}$ as an equivariant spectrum over the Galois group. As above, we will write $\text{Gal} = \text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$, so that $\mathbb{G}_n \cong \mathbb{S}_n \rtimes \text{Gal}$.

We begin with the following elementary fact: the map $\mathbb{Z}_p \rightarrow \mathbb{W}$ is Galois with Galois group Gal ; thus, it is faithfully flat, étale, and the shearing map

$$\mathbb{W} \otimes_{\mathbb{Z}_p} \mathbb{W} \rightarrow \text{map}(\text{Gal}, \mathbb{W})$$

sending $a \otimes b$ to the function $g \mapsto ag(b)$ is an isomorphism. In fact, this shearing map is certainly an isomorphism modulo p ; then the statement follows from Nakayama's Lemma. Faithfully flat descent now implies that the category of \mathbb{Z}_p -modules is equivalent to the category of twisted $\mathbb{W}[\text{Gal}]$ -modules under the functor $M \mapsto \mathbb{W} \otimes_{\mathbb{Z}_p} M$; the inverse to this functor sends N to N^{Gal} .

This extends to the following result.

Lemma 1.24. *Let $K \subseteq \mathbb{G}_n$ be a closed subgroup and let $K_0 = K \cap \mathbb{S}_n$. Suppose the canonical map*

$$K/K_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \text{Gal}$$

is an isomorphism. Then for any twisted \mathbb{G}_n -module M we have isomorphisms

$$\begin{aligned} H^*(K, M) &\cong H^*(K_0, M)^{\text{Gal}} \\ H^*(K_0, M) &\cong \mathbb{W} \otimes_{\mathbb{Z}_p} H^*(K, M) . \end{aligned}$$

Proof. The subgroup \mathbb{S}_n acts on E_0 through \mathbb{W} -algebra homomorphisms; hence it acts on M through \mathbb{W} -module homomorphisms. It follows that we can write the functor $(-)^K$ of invariants as a composite functor

$$\text{twisted } \mathbb{G}_n\text{-modules} \xrightarrow{(-)^{K_0}} \text{twisted } \mathbb{W}[\text{Gal}]\text{-modules} \xrightarrow{(-)^{\text{Gal}}} \mathbb{Z}_p\text{-modules}.$$

As we just remarked, the second of these two functors is an equivalence of categories and in particular it is an exact functor. The first equation follows. The second equation follows from the first and the fact that the inverse to $(-)^{\text{Gal}}$ is the functor $M \mapsto \mathbb{W} \otimes_{\mathbb{Z}_p} M$. \square

We now give a fact seemingly known to everyone, but hard to find in print.

Lemma 1.25. *For all p and all $n \geq 1$ we have isomorphisms*

$$\begin{aligned} H^0(\mathbb{G}_n, E_0) &\cong \mathbb{Z}_p \\ H^0(\mathbb{S}_n, E_0) &\cong \mathbb{W} = W(\mathbb{F}_{p^n}). \end{aligned}$$

Furthermore, $H^0(\mathbb{G}_n, E_t) = H^0(\mathbb{S}_n, E_t) = 0$ if $t \neq 0$.

Proof. By Lemma 1.24 we need only do the case of \mathbb{G}_n . Since E is Landweber exact, we have (before $K(n)$ -localization)

$$\pi_*(E \wedge E) \cong E_* \otimes_{MU_*} MU_* MU \otimes_{MU_*} E_*$$

and $\pi_*(E \wedge E)$ is a flat E_* -module. The map $\eta_L \otimes \eta_R : MU_* \otimes MU_* \rightarrow MU_* MU$ is a rational isomorphism; hence,

$$\eta_L \otimes \eta_R : E_* \otimes_{\mathbb{Z}_p} E_* \longrightarrow \pi_*(E \wedge E)$$

is also a rational isomorphism and an injection, as the source is torsion free.

Since $\pi_*(E \wedge E)$ is a flat E_* -module, the map $\pi_*(E \wedge E) \rightarrow \pi_* L_{K(n)}(E \wedge E)$ is \mathfrak{m} -adic completion and remains an injection by [HS99, Proposition 2.5]. Composing this map with $\eta_L \otimes \eta_R$ yields an injection

$$i : E_* \otimes_{\mathbb{Z}_p} E_* \longrightarrow \pi_* L_{K(n)}(E \wedge E) \cong \text{map}(\mathbb{G}_n, E_*)$$

under the shearing map sending $a \otimes b$ to the function $g \mapsto ag(b)$. Thus if $a \in E_*$ and $g(a) = a$ for all $g \in \mathbb{G}_n$, we have $a \otimes 1 - 1 \otimes a$ goes to zero under the map i . If $t = 0$, then $a \in \mathbb{Z}_p \subseteq E_0$. If $t \neq 0$, then $a = 0$. \square

Remark 1.26. In the proof of Lemma 1.27 below we will use the following observation. Suppose we have a spectral sequence $\{E_r^{s,t}\}$ that is multiplicative; that is, $E_r^{*,*}$ is a bigraded ring which is commutative up to sign and d_r satisfies the Leibniz rule, again up to sign. Further suppose $R \subseteq E_r^{0,0}$ is a commutative subring of d_r -cycles and $R \subseteq S$ is an étale extension in $E_r^{0,0}$. Then every element of S is a d_r -cycle. To see this, note that d_r restricted to S is a derivation over R and any such derivation must vanish; indeed, depending on your foundations, the vanishing of such derivations may even be part of your definition of étale.

Lemma 1.27. *For all p and all $n \geq 1$ there is a Gal-equivariant equivalence*

$$\text{Gal}^+ \wedge L_{K(n)} S^0 \rightarrow E^{h\mathbb{S}_n}.$$

Proof. We have an injection $\mathbb{W} \rightarrow \pi_0 E^{h\mathbb{S}_n}$. To see this, we begin with the isomorphism $\mathbb{W} \cong H^0(\mathbb{S}_n, E_0)$ of Lemma 1.25. Since $\mathbb{Z}_p \subseteq \mathbb{W}$ is an étale extension and the Adams-Novikov Spectral Sequence for \mathbb{S}_n is a spectral sequence of rings, all of \mathbb{W} survives to E_∞ and the edge homomorphism provides a surjection $\pi_0 E^{h\mathbb{S}_n} \rightarrow \mathbb{W}$ of rings. The kernel has only nilpotent elements because the spectral sequence has a horizontal vanishing line at E_∞ . (See Remark 1.14 for more on this point.) The unit map gives a splitting, as rings, of the inclusion $\mathbb{Z}_p \rightarrow \mathbb{W}$ to $\pi_0 E^{h\mathbb{S}_n}$. Again since $\mathbb{Z}_p \subseteq \mathbb{W}$ is étale, this lifting extends uniquely to \mathbb{W} . This yields the injection we need.

Now define $\omega : S^0 \rightarrow E^{h\mathbb{S}_n}$ to be a representative of the homotopy class defined by a primitive $(p^n - 1)$ st root of unity in \mathbb{W} . We can extend ω to a Gal-equivariant

map $f : \text{Gal}^+ \wedge S^0 \rightarrow E^{h\mathbb{S}_2}$ inducing the splitting $\mathbb{W} \rightarrow \pi_0 E^{h\mathbb{S}_2}$; here we use that the map $\mathbb{Z}_p[\text{Gal}] \rightarrow \mathbb{W}$

$$\sum a_g g \mapsto \sum a_g g(\omega)$$

is an isomorphism. The map f extends to an isomorphism of twisted \mathbb{G}_n -modules

$$E_* f : E_*(\text{Gal}^+ \wedge S^0) \cong \text{map}(\text{Gal}, E_*) \cong \text{map}(\mathbb{G}_n/\mathbb{S}_n, E_*) \cong E_* E^{h\mathbb{S}_n}$$

thus completing the argument. \square

The following result is a topological analog of Lemma 1.24.

Lemma 1.28. *Let $K \subseteq \mathbb{G}_n$ be a closed subgroup and let $K_0 = K \cap \mathbb{S}_n$. Suppose the canonical map*

$$K/K_0 \longrightarrow \mathbb{G}_n/\mathbb{S}_n \cong \text{Gal}$$

is an isomorphism. Then there is a Gal-equivariant equivalence

$$\text{Gal}^+ \wedge E^{hK} \rightarrow E^{hK_0}.$$

Proof. This follows from Lemma 1.27. Define a map $E^{h\mathbb{S}_n} \wedge E^{hK} \rightarrow E^{hK_0}$ by the composition

$$(1.29) \quad E^{h\mathbb{S}_n} \wedge E^{hK} \rightarrow E^{hK_0} \wedge E^{hK_0} \rightarrow E^{hK_0}$$

where the first map is given by the inclusion and the last map is multiplication. In E^* -cohomology, this produces the map

$$E^*[[\mathbb{G}_n/K_0]] \rightarrow E^*[[\mathbb{G}_n/\mathbb{S}_n]] \otimes_{E^*} E^*[[\mathbb{G}_n/K]]$$

induced by the maps on cosets

$$\mathbb{G}_n/K_0 \rightarrow \mathbb{G}_n/K_0 \times \mathbb{G}_n/K_0 \rightarrow \mathbb{G}_n/\mathbb{S}_n \times \mathbb{G}_n/K$$

where the first map is the diagonal and the second map is projection. Since $K/K_0 \cong \mathbb{G}_n/\mathbb{S}_n$, this map on cosets is an isomorphism; therefore, the map of (1.29) is an E^* -isomorphism. Since $E_* E^{hG}$ is a profree E_* -module for all closed G , the map is an E_* -isomorphism, hence a weak equivalence by Remark 1.14. \square

Remark 1.30. Combining Lemma 1.24 and Lemma 1.28 yields an isomorphism of spectral sequences

$$\begin{array}{ccc} \mathbb{W} \otimes H^*(K, E_*) & \Longrightarrow & \mathbb{W} \otimes \pi_* E^K \\ \cong \downarrow & & \downarrow \cong \\ H^*(K_0, E_*) & \Longrightarrow & \pi_* E^{hK_0} \end{array}$$

where the differentials on the top line are the \mathbb{W} -linear differentials extended from the spectral sequence for K .

Remark 1.31. At $n = 2$ and $p = 2$, Lemma 1.28 applies to the case of $K = G_{48}$; then $K_0 = G_{24}$. This implies that any of the spectra

$$X(i, j) \stackrel{\text{def}}{=} \Sigma^{24i} E^{hG_{48}} \vee \Sigma^{24j} E^{hG_{48}}$$

has the property that $E_* X(i, j) \cong E_* E^{hG_{24}}$. But $X(i, j) = \Sigma^{24i} E^{hG_{24}}$ if and only if $i \equiv j \pmod{8}$. This means that in some of the arguments we give to prove our main result we will have to produce two homotopy classes rather than one. See Theorem 5.8.

2. THE HOMOTOPY GROUPS OF HOMOTOPY FIXED POINT SPECTRA

Here we collect what we will need about the homotopy groups of E^{hF} , where F runs through the finite subgroups of \mathbb{G}_2^1 of Remark 1.11. We will be working entirely at $n = p = 2$ and using the formal group from the supersingular curve of (1.7). Much of what's needed is in the literature and we'll do our best to give references. However, much of what is written is for calculations over Hopf algebroids, which is not quite what we're doing, and the results need translation. In addition, many of the results as written include some variant of the phrase "we neglect the *bo*-patterns". We make this thought precise with the following *ad hoc* definition.

Definition 2.1. Let $F \subseteq \mathbb{G}_2$ be any finite subgroup containing $C_2 = \{\pm 1\}$. Then we define the ***bo*-patterns** $L_1(\pi_* E^{hF})$ of $\pi_* E^{hF}$ to be the image of the map in homotopy

$$\pi_* E^{hF} \longrightarrow \pi_* L_{K(1)} E^{hF}.$$

We also define the **pure $K(2)$ -classes** $M_2(\pi_* E^{hF})$ to be the kernel of the same map.

Thus we have a short exact sequence

$$0 \rightarrow M_2(\pi_* E^{hF}) \rightarrow \pi_* E^{hF} \rightarrow L_1(\pi_* E^{hF}) \rightarrow 0.$$

Notice that the *bo*-patterns are defined as a quotient. In most cases, this sequence is not split as modules over the homotopy groups of spheres.

Remark 2.2. The name *bo*-patterns is something of a misnomer, as *KO*-patterns would more accurately. Here *KO* is 8-periodic 2-complete real *K*-theory. In all our examples we will have an isomorphism

$$R(F) \otimes_{\mathbb{Z}_2} KO_* \cong \pi_* L_{K(1)} E^{hF}$$

for some \mathbb{Z}_2 -algebra $R(F)$ in degree zero. While $R(F) \otimes_{\mathbb{Z}_2} KO_*$ is 8-periodic, $L_1(\pi_* E^{hF})$ will typically have $8k$ -periodicity for some $k > 1$. As a warning, we mention that this isomorphism is simply as rings; we are not claiming $L_{K(1)} E^{hF}$ is a *KO*-algebra.

Remark 2.3. Here is more detail, to explain our thinking.

Let us write $S/2^n$ for the $\mathbb{Z}/2^n$ -Moore spectrum. Then there is a weak equivalence

$$L_{K(1)} X \simeq \text{holim } v_1^{-1}(X \wedge S/2^n)$$

and, if X is $K(2)$ -local, a corresponding localized Adams-Novikov Spectral Sequence

$$(2.4) \quad \lim v_1^{-1} H^*(\mathbb{G}_2, (E_* X)/2^n) \Longrightarrow \pi_* L_{K(1)} X.$$

This spectral sequence doesn't obviously converge.

Now suppose $C_2 = \{\pm 1\} \subseteq F \subset G_{48}$. Then the spectral sequence (2.4) for $X = E^{hF}$ becomes

$$(2.5) \quad \lim v_1^{-1} H^*(F, (E_*)/2^n) \Longrightarrow \pi_* L_{K(1)} E^{hF}.$$

Using Strickland's formulas [Str] it is possible to show that

$$\lim v_1^{-1} H^*(F, (E_*)/2^n) \cong \lim v_1^{-1} H^*(C_2, (E_*)/2^n)^{F/C_2}$$

and that

$$v_1^{-1}H^*(C_2, (E_*)/2^n) \cong \mathbb{W}((u_1))[u^{\pm 2}, \eta]$$

where $\mathbb{W}((u_1)) = \lim(\mathbb{W}/2^n)[[u_1^{\pm 1}]]$ and $\eta \in H^1(C_2, E_2)$ detects the class of the same name in $\pi_1 S^0$. (See (2.6) below.) From this it follows that the spectral sequence (2.5) is completely determined by the standard differential $d_3(v_1^2) = \epsilon\eta^3$, where $v_1^2 = u_1^2 u^{-2}$ and $\epsilon \in \mathbb{F}_4((u_1))^\times$ is a unit. We can conclude that the spectral sequence converges and

$$\pi_* L_{K(1)} E^{hF} \cong \mathbb{W}((u_1))^F \otimes_{\mathbb{Z}_2} KO_*$$

and in particular, that $\pi_* L_{K(1)} E^{hF}$ and $L_1(\pi_* E^{hF})$ are both concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8. It then remains to analyze the pure $K(2)$ -local classes.

Now, not much of what we just wrote is explicitly in print, and it would take quite a few pages to prove in detail. But we will put together what we can from the existing literature to cover the main points case-by-case below. See Propositions 2.7, 2.11, and 2.17

2.1. The homotopy groups of E^{hC_2} and E^{hC_6} . The standard source here is Mahowald-Rezk [MR09], which uses the Hopf algebroid approach, but also uses the same elliptic curve we have chosen. So the translation is straightforward. Here is a summary.

The central $C_2 \subseteq \mathbb{G}_2$ acts trivially on E_0 and by multiplication by -1 on u ; hence

$$(2.6) \quad H^*(C_2, E_*) \cong \mathbb{W}[[u_1]][u^{\pm 2}, \alpha]/(2\alpha)$$

where $\alpha \in H^1(C_2, E_2)$ is the image of the generator of $H^1(C_2, \mathbb{Z}\langle \text{sgn} \rangle)$ under the map which sends the generator of the sign representation to u^{-1} . Since

$$v_1 = u_1 u^{-1} \in H^0(C_2, E_2/2)$$

and the class $u_1 \alpha \in H^1(C_2, E_2)$ is the image of v_1 under the integral Bockstein, the class $\eta \in \pi_1 S^0$ is detected by $u_1 \alpha$. We will also write $\eta = u_1 \alpha$.

Proposition 2.7. *The class $b(C_2) \stackrel{\text{def}}{=} u_1^2 u^{-2}$ reduces to v_1^2 in $v_1^{-1}H^*(C_2, E_*/2)$. There is an isomorphism*

$$\mathbb{W}((u_1))[b(C_2)^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1}H^*(C_2, E_*/2^n).$$

The standard differential $d_3(v_1^2) = \eta^3$ (see Lemma 2.21 below) forces a differential

$$d_3(u^{-2}) = \epsilon u_1 \alpha^3$$

where $\epsilon \in \mathbb{F}_3[[u_1]]^\times$. Using the Mahowald-Rezk transfer argument [MR09, Prop. 3.5] we have $\nu \in \pi_3 S^0$ is non-zero in $\pi_3 E^{hC_2}$ and detected by α^3 ; this in turn forces a differential

$$d_7(u^{-4}) = \alpha^7 = \alpha \nu^2.$$

The spectral sequence collapses at E_7 and we have the following result.

Proposition 2.8. *The homotopy ring $\pi_* E^{hC_2}$ is periodic of period 16 with periodicity generator e_{16} detected by u^{-8} . The bo-patterns $L_1(E^{hC_2})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$ -local classes $M_2(E^{hC_2})$ is generated by the classes*

$$\alpha^i e_{16}^k, \quad k \in \mathbb{Z}, \quad i = 3, 4, 5, 6.$$

To get the homotopy of E^{hC_6} , with $C_6 = C_2 \times \mathbb{F}_4^\times$, we need to know that action of \mathbb{F}_4^\times . We can use Strickland's calculations [Str] or interpret the Mahowald-Rezk results. Let $\omega \in \mathbb{F}_4^\times$ be a primitive cube root of unity. Then, $\omega_* u = \omega u$ and $\omega_* u_1 = \omega u_1$; it follows that $\omega_* \alpha = \omega^{-1} \alpha$. The next result can be deduced from these formulas and the fact that

$$\pi_* E^{hC_6} \cong (\pi_* E^{hC_2})^{h\mathbb{F}_4^\times}.$$

Proposition 2.9. *The homotopy ring $\pi_* E^{hC_6}$ is periodic of period 48 with periodicity generator e_{16}^3 detected by u^{-24} . The bo-patterns $L_1(E^{hC_6})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$ -local classes $M_2(E^{hC_6})$ is generated by the classes*

$$e^{3k} \alpha^3 \quad e^{3k} \alpha^6 \quad e^{3k+1} \alpha^4 \quad e^{3k+2} \alpha^5$$

of degrees $48k + 3$, $48k + 6$, $48k + 20$, and $48k + 37$ respectively. Furthermore, the homotopy class $\nu \in \pi_3 S^0$ is detected by the class α^3 and the class $\bar{\kappa} \in \pi_{20} S^0$ is detected by $e_{16} \alpha^4$.

2.2. The homotopy groups of E^{hC_4} . The standard reference for this calculation is Behrens-Ormsby [BO16], and again they use Hopf algebroids. Here we hit another small problem: the supersingular elliptic curve they use is different from the one we have chosen, and thus they have a different action of C_4 on a different version of Morava E -theory. There are two possible solutions. One is to do the calculations over again, using Strickland's formulas. The other is to notice that the two supersingular curves become isomorphic over the algebraically closed field $\overline{\mathbb{F}}_2$ and to use descent to make the calculations. Using either method we obtain the following result. Let $i \in C_4$ be a generator and let

$$z = u_1 + i_* u_1 \in H^0(C_4, E_0).$$

There are further cohomology classes

$$b_2 \in H^0(C_4, E_4) \quad \delta \in H^0(C_4, E_8)$$

and

$$\gamma \in H^1(C_4, E_6) \quad \xi \in H^2(C_4, E_8).$$

Let $\eta \in H^1(C_4, E_2)$ and $\nu \in H^1(C_4, E_4)$ be the images of the like-named classes from the BP -based Adams-Novikov Spectral Sequence (see Remark 1.13).

Proposition 2.10. *There is an isomorphism*

$$H^*(C_4, E_*) \cong \mathbb{W}[[z]][b_2, \delta^{\pm 1}, \eta, \nu, \gamma, \xi]/R$$

where R is the ideal of relations given by

$$2\eta = 2\gamma = 4\xi = 0$$

and

$$b_2^2 \equiv z^2 \sigma_2 \quad \text{mod } 2$$

and

$$\delta\eta^2 = b_2\xi = \gamma^2 \quad b_2\gamma = z\sigma_2\eta$$

and

$$b_2\eta = z\gamma \quad \gamma\eta = z\xi$$

and the final relations involving ν :

$$\nu^2 = 2\xi \quad \eta\nu = b_2\nu = \gamma\nu = 0$$

This result is displayed in Figure 1 below, presented in the standard Adams format: the x -axis is $t-s$; the y -axis is s . In this chart, the square box \square represents a copy of $\mathbb{W}[[z]]$, the circle \circ a copy of $\mathbb{F}_4[[z]]$, and the crossed circle \otimes a copy of $\mathbb{W}[[z]]/(4, 2z)$ generated by a class of the form $\xi^j\delta^i$. A solid dot is a copy of \mathbb{F}_4 annihilated by z ; it is generated by a class of the form $\xi^j\nu$. The solid lines are multiplication by η or ν , as needed, and a dashed line indicates that $x\eta = zy$, where x and y are generators in the appropriate bidegree.

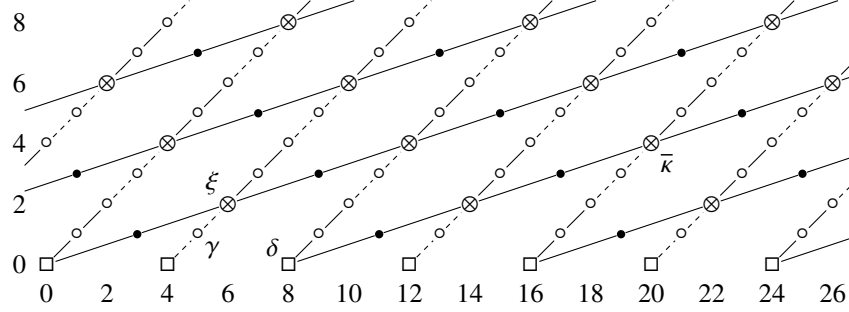


FIGURE 1. The cohomology of C_4

Notice that $H^*(C_4, E_*)$ is 8-periodic with the periodicity class δ and that

$$\times \xi : H^s(C_4, E_*) \rightarrow H^{s+2}(C_4, E_{*+8})$$

is onto for $s \geq 0$ and an isomorphism for $s > 0$. In fact $\delta^{-1}\xi \in H^2(C_4, E_0)$ is, up to multiplication by a unit, the image of the periodicity class for the group cohomology of C_4 under the inclusion of trivial coefficients:

$$\mathbb{Z}/4 \cong H^4(C_4, \mathbb{Z}_2) \cong H^2(C_4, \mathbb{W}) \rightarrow H^s(C_4, E_0).$$

Proposition 2.11. *Modulo 2 we have an equivalence $b_2 \equiv v_1^2$ and then an isomorphism*

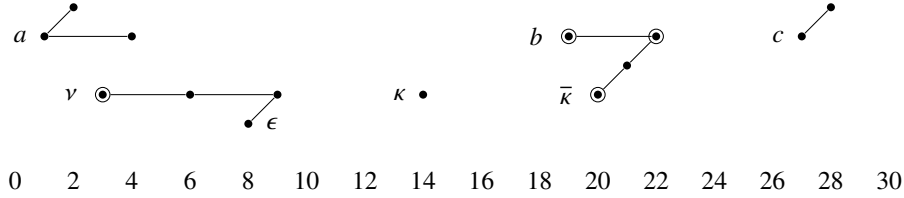
$$\mathbb{W}((z))[b_2^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1} H^*(C_4, E_*/2^n).$$

The differentials and extensions in this spectral sequence go exactly as in the paper by Behrens and Ormsby [BO16] and we end with the following result.

Proposition 2.12. *The homotopy ring $\pi_* E^{hC_4}$ is periodic of period 32 with periodicity generator e_{32} detected by δ^4 . The bo-patterns $L_1(E^{hC_4})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8 and the group of pure $K(2)$ -local classes $M_2(E^{hC_4})$ is generated by the classes $e_{32}^k x$ where x is from the following table*

<i>Class</i>	<i>Degree</i>	<i>Order</i>	<i>E₂-name</i>
a	1	2	$\delta^{-1}\xi\nu$
$a\eta$	2	2	$\delta^{-2}\xi^2\nu^2$
ν	3	4	ν
$a\nu$	4	2	$\delta^{-1}\xi\nu^2$
ν^2	6	2	ν^2
ϵ	8	2	$\delta^{-1}\xi^4$
$\nu^3 = \eta\epsilon$	9	2	$\delta^{-1}\xi^4\eta$
κ	14	2	$\delta\nu^2$
b	19	4	$\delta^2\nu$
$\bar{\kappa}$	20	4	$\delta\xi^2$
$\eta\bar{\kappa}$	21	2	$\eta\xi^3$
$b\nu$	22	4	$\delta^2\nu^2$
c	27	2	$\delta^2\gamma\xi$
$c\eta$	28	2	$\delta^{-1}\xi^6$

The pure $K(2)$ classes in $\pi_*E^{hC_4}$ are presented in the Figure 2: the horizontal bar is multiplication by ν , the diagonal bar is multiplication by η . Note that $\bar{\kappa}\eta^2 = 2b\nu$. The horizontal scale is degree of the element, but the vertical scale has no meaning. Many of the additive and multiplicative relations are given by exotic extensions in this spectral sequence and the meaning of the original Adams-Novikov filtration becomes attenuated as a result; see Behrens-Ormsby [BO16] for details.


 FIGURE 2. The pure $K(2)$ classes in $\pi_*E^{hC_4}$

2.3. The homotopy groups of $E^{hG_{24}}$ and $E^{hG_{48}}$. Remarks 1.30 and 1.31 yield an isomorphism of spectral sequences

$$\begin{array}{ccc}
 \mathbb{W} \otimes_{\mathbb{Z}_2} H^*(G_{48}, E_*) & \Longrightarrow & \mathbb{W} \otimes \pi_* E^{hG_{48}} \\
 \cong \downarrow & & \cong \downarrow \\
 H^*(G_{24}, E_*) & \Longrightarrow & \pi_* E^{hG_{24}} .
 \end{array}$$

Therefore, we focus on the case of G_{48} . Here the standard sources are [Bau08], [DFHH14] and [HM98] although it requires some translation in each case to get the results we want.

The ring $H^0(G_{48}, E_*)$ is isomorphic to the ring of modular forms for supersingular elliptic curves at the prime 2. Then there are elements

$$c_4 \in H^0(G_{48}, E_8) \quad c_6 \in H^0(G_{48}, E_{12}) \quad \Delta \in H^0(G_{48}, E_{24})$$

obtained from the modular forms of the same name for our supersingular curve. Since this curve is smooth, Δ is invertible and the j -invariant of our curves $j = c_4^3/\Delta \in H^0(G_{48}, E_0)$ is defined. Then we get an isomorphism

$$\mathbb{Z}_2[[j]][c_4, c_6, \Delta^{\pm 1}]/(c_4^2 - c_6^2 = (12)^3 \Delta, \Delta j = c_4^3) \cong H^0(G_{48}, E_*).$$

Modulo 2 we get a slightly simpler answer:

$$\mathbb{F}_2[[j]][v_1, \Delta^{\pm 1}]/(j\Delta = v_1^{12}) \cong H^0(G_{48}, E_*/2).$$

Modulo 2 we have congruences

$$(2.13) \quad c_4 \equiv v_1^4 \quad \text{and} \quad c_6 \equiv v_1^6.$$

To describe the higher cohomology, we make a table of multiplicative generators. For each x , the bidegree of x is (s, t) if $x \in H^s(G_{48}, E_t)$. All but μ detect the elements of the same name in $\pi_* S^0$. Furthermore, all elements but $\bar{\kappa}$ are in the image of the map (see Remark 1.13)

$$\text{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*) \rightarrow H^*(\mathbb{G}_2, E_*) \rightarrow H^*(G_{48}, E_*).$$

Hence we also give the name (the “MRW” is for Miller-Ravenel-Wilson) of a preimage. The Greek letter notation is that of [MRW77].

Class	Bidegree	Order	MRW
η	$(1, 2)$	2	α_1
ν	$(1, 4)$	4	$\alpha_{2/2}$
μ	$(1, 6)$	2	α_3
ϵ	$(2, 10)$	2	β_2
κ	$(2, 16)$	2	β_3
$\bar{\kappa}$	$(4, 24)$	8	—

The class $\bar{\kappa} \in \pi_{20} S^0$ is detected by the image of β_4 in $H^2(\mathbb{G}_2, E_*)$. The class μ has a special role which we discuss below in Lemma 2.21, but we would like to note right away that

$$(2.14) \quad v_1^2 \eta \equiv \mu \quad \text{modulo } 2.$$

The following result is actually much easier to visualize than to write down. See the Figure 3 below.

Theorem 2.15. *There is an isomorphism*

$$H^0(G_{48}, E_*)[\eta, \nu, \mu, \epsilon, \kappa, \bar{\kappa}]/R \cong H^*(G_{48}, E_*)$$

where R is the ideal defined by

(1) the relations for ν :

$$\eta\nu = 2\nu^2 = \nu^4 = \mu\nu = 0;$$

(2) the relations for ϵ :

$$\eta\epsilon = \nu^3, \quad \nu\epsilon = \epsilon^2 = \mu\epsilon = 0;$$

(3) the relations for κ :

$$\nu^2\kappa = 4\bar{\kappa}, \quad \eta^2\kappa = \epsilon\kappa = \kappa^2 = \mu\kappa = 0;$$

(4) the elements annihilated by modular forms

$$c_4\nu = c_6\nu = c_4\epsilon = c_6\epsilon = c_4\kappa = c_6\kappa = 0;$$

(5) the relations between $\bar{\kappa}$ and modular forms;

$$c_4\bar{\kappa} = \Delta\eta^4, \quad c_6\bar{\kappa} = \Delta\eta^3\mu \quad \mu^2\bar{\kappa} = \Delta\eta^6;$$

(6) and the relations indicated by the congruences of (2.13) and (2.14):

$$\mu^2 = c_4\eta^2 \quad c_4\mu = c_6\eta \quad c_6\mu = c_4^2\eta.$$

This result is presented graphically in Figure 3 below. We present it as the E_2 page of the Adams-Novikov Spectral Sequence. The cohomology is 24-periodic on Δ , and the spectral sequence fills the entire upper- half plane. In Figure 3, the square box \square represents a copy of $\mathbb{Z}_2[[j]]$, the circle \circ a copy of $\mathbb{F}_2[[j]]$, and the crossed circle \otimes a copy of $\mathbb{Z}_2[[j]]/(8, 2j)$ generated by a class of the form $\Delta^i\bar{\kappa}^j$. The solid bullet represents a class of order 2 annihilated by j and the doubled bullet a class of order 4 annihilated by j ; these last classes are always of the form $\Delta^i\bar{\kappa}^j\nu$. The solid lines are multiplication by η or ν , as needed, and a dashed line indicates that $x\eta = jy$, where x and y are generators in the appropriate bidegree.

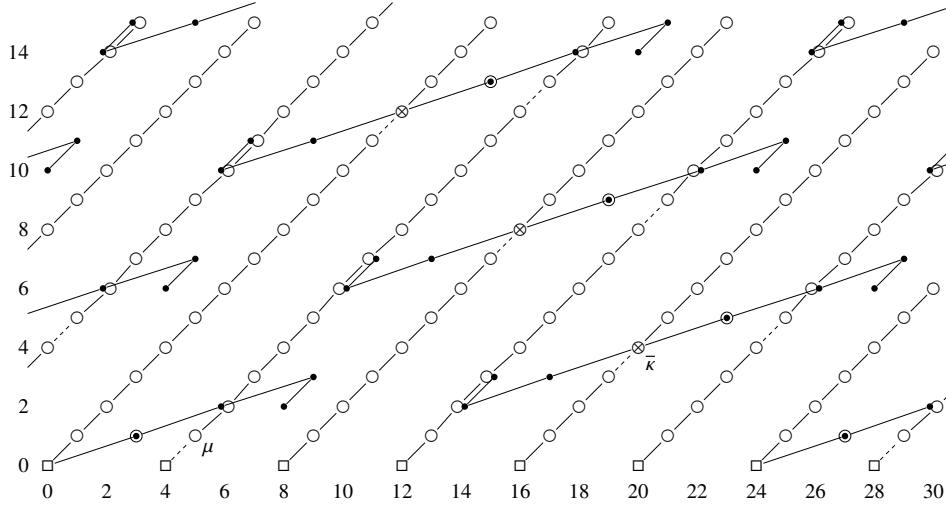


FIGURE 3. The cohomology of G_{48}

Remark 2.16. (1) Many of the later relations can be rephrased as relations for multiplication by $j = c_4^3/\Delta$. For example Theorem 2.15 (4) implies

$$j\nu = j\epsilon = j\kappa = 0$$

and (5) implies

$$j\bar{\kappa} = c_4^2\eta^4$$

and (6) implies

$$j\mu = c_4^2 c_6 \Delta^{-1} \eta.$$

These last two equations explain the dashed lines in Figure 4.

(2) Multiplication by $\bar{\kappa} : H^s(G_{48}, E_t) \rightarrow H^{s+4}(G_{48}, E_{t+24})$ is surjective and an isomorphism if $s > 0$. In fact, up to a unit, $\Delta^{-1}\bar{\kappa} \in H^4(G_{48}, E_0)$ is the image of the periodicity class in group cohomology for Q_8 under the inclusion of trivial coefficients:

$$\mathbb{Z}/8 \cong H^4(Q_8, \mathbb{W})^{G_{48}/Q_8} \cong H^4(G_{48}, \mathbb{W}) \rightarrow H^4(G_{48}, E_0).$$

The congruence (2.13) and the relations of Theorem 2.15 now give the following result. Note that the class c_4 becomes invertible in $\lim v_1^{-1} H^*(G_{48}, E_*/2^n)$ and we may define $b_2 = c_6/c_4$. (**Warning:** This class b_2 is related to, but not quite the same, as the class b_2 of Proposition 2.10. Both uses of b_2 appear in the literature.)

Proposition 2.17. *The class b_2 reduces to v_1^2 in $v_1^{-1} H^*(G_{48}, E_*/2)$. There are isomorphisms*

$$\mathbb{Z}_2((j))[b_2^{\pm 1}, \eta]/(2\eta) \cong \lim v_1^{-1} H^*(G_{48}, E_*/2^n).$$

and

$$\mathbb{F}_2((j))[v_1^{\pm 1}, \eta] \cong v_1^{-1} H^*(G_{48}, E_*/2).$$

Under the reduction map $H^*(G_{48}, E_*) \rightarrow H^*(G_{48}, E_*/2)$ we have

$$\begin{aligned} c_4 &\mapsto v_1^4 \\ c_6 &\mapsto v_1^6 \\ c_4 \bar{\kappa} &\mapsto v_1^4 \bar{\kappa} = \Delta \eta^4 \\ \mu &\mapsto v_1^2 \eta. \end{aligned}$$

Under the localization map $H^*(G_{48}, E_*) \rightarrow v_1^{-1} H^*(G_{48}, E_*/2)$ we have

$$\begin{aligned} \Delta &\mapsto v_1^{12}/j \\ \bar{\kappa} &\mapsto v_1^8 \eta^4/j \end{aligned}$$

and that ν , ϵ , and κ map to zero.

We have the following; see [HM98], [Bau08], or [DFHH14].

Proposition 2.18. *The homotopy ring $\pi_* E^{hG_{48}}$ is periodic of period 192 with periodicity generator detected by Δ^8 . The bo-patterns $L_1(E^{hG_{48}})$ are concentrated in degrees congruent to 0, 1, 2, and 4 modulo 8.*

Remark 2.19. We will not try to enumerate the pure $K(2)$ -classes of $M_2(E^{hG_{48}})$; this information is known (by the same references as for Proposition 2.18), but we won't need that information in its entirety and it is rather complicated to write down. What we will need can be read off of Figure 4, which is adapted from the charts created by Tilman Bauer [Bau08], Section 8.

This chart shows a section of the E_∞ -page of the Adams-Novikov Spectral Sequence

$$H^s(G_{48}, E_t) \implies \pi_{t-s} E^{hG_{48}}.$$

It is in the standard Adams bigrading $(t-s, s)$.

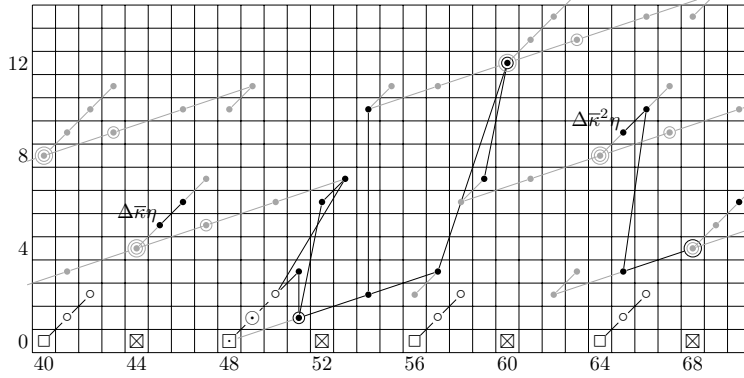


FIGURE 4. The homotopy groups $\pi_i E^{hG_{48}}$ for $40 \leq i \leq 70$.

Some additive and multiplicative extensions are displayed as well. The non-zero permanent cycles are in black; some other elements, mostly built from patterns around elements of the form $\Delta^j \kappa^i$, have been left in gray for orientation, even though they do not last to the E_∞ -page. Bullets with circles are elements of order 4; bullets with two circles are elements of order 8. Vertical lines are extensions by multiplication by 2, lines raising homotopy degree by 1 are η -extensions, lines raising homotopy degree by 3 are ν -extensions.

The lines $0 \leq s \leq 2$ display the bo -patterns; the adorned boxes and circles all represent ideals of either $\mathbb{Z}_2[[j]]$ or $\mathbb{F}_2[[j]]$:

$$\begin{aligned} \square &\cong \mathbb{Z}_2[[j]] \\ \boxtimes &\cong (2) \subseteq \mathbb{Z}_2[[j]] \\ \boxdot &\cong (4, j) \subseteq \mathbb{Z}_2[[j]] \\ \circ &\cong \mathbb{F}_2[[j]] \\ \odot &\cong (j) \subseteq \mathbb{F}_2[[j]]. \end{aligned}$$

Elements not falling into one of these patterns are annihilated by j . The η -extension from $(t-s, s) = (65, 3)$ entry is ambiguous. We mark it as non-zero because we may choose, as Bauer does, the two generators of the group of pure $K(2)$ -classes in $\pi_{65} E^{hG_{48}}$ to be

$$e[45, 5]\bar{\kappa} \quad \text{and} \quad e[51, 1]\kappa$$

where $e[45, 5] \in \pi_{45} E^{hG_{48}}$ and $e[51, 1] \in \pi_{51} E^{hG_{48}}$ are generators detected by $\Delta \bar{\kappa} \eta$ and $\Delta^2 \nu$ respectively. The class $e[51, 1]\kappa$ is detected on the $s = 3$ line by $\Delta^2 \kappa \nu$ and $e[51, 1]\kappa \eta \neq 0$.

We now record, from Figure 4, some data about our crucial homotopy classes.

Lemma 2.20. *There is an isomorphism*

$$\mathbb{Z}/2 \cong \pi_{45} E^{hG_{48}}.$$

The generator is detected by the class

$$\Delta \bar{\kappa} \eta \in H^5(G_{48}, E_{50}).$$

The class $\Delta \bar{\kappa}^2 \eta^2 \in H^{10}(G_{48}, E_{76})$ is a non-zero permanent cycle detecting a generator of the subgroup $\pi_{66} E^{hG_{48}}$ of the pure $K(2)$ -classes of that degree.

We close with some remarks on the role of μ in the d_3 differentials.

Lemma 2.21. *Let $\mu \in H^1(\mathbb{G}_2, E_6)$ be the image of the class*

$$\alpha_3 \in \text{Ext}_{BP_*BP}^1(\Sigma^6 BP_*, BP_*).$$

Then in any of the Adams-Novikov Spectral Sequences

$$H^*(F, E_*) \Longrightarrow \pi_{t-s} E^{hF}$$

and for any $x \in H^(F, E_*)$ we have*

$$d_3(x\mu) = d_3(x)\mu + x\eta^4.$$

In the spectral sequences

$$(2.22) \quad H^*(F, E_*/2) \Longrightarrow \pi_{t-s}(E^{hF} \wedge S/2)$$

we have

$$d_3(v_1^2 x) = d_3(x)v_1^2 + x\eta^3 + y$$

where $y\eta = 0$. Finally, in the spectral sequence (2.22) we have

$$d_3(v_1^4 x) = v_1^4 d_3(x).$$

Proof. In the Adams-Novikov spectral sequence

$$\text{Ext}_{BP_*BP}^s(\Sigma^t BP_*, BP_*) \Longrightarrow \mathbb{Z}_{(2)} \otimes \pi_{t-s} S^0.$$

we have $d_3(\alpha_3) = \eta^4$. (In fact, by [MRW77], Corollary 4.23, $\eta^4 \neq 0$ at E_2 and $E_2^{1,6} \cong \mathbb{Z}/2$ generated by α_3 . The differential is then forced.) Since the fixed point spectral sequences and the localizations are modules over this standard Adams-Novikov Spectral Sequence, the first formula follows. The second formula follows because $v_1^2 \eta = \alpha_3$ in

$$\text{Ext}_{BP_*BP}^s(\Sigma^t BP_*, BP_*/2).$$

The third formula follows from the fact that $S/2$ has a v_1^4 -self map. \square

3. ALGEBRAIC AND TOPOLOGICAL RESOLUTIONS

In this section we review the centralizer resolution constructed by Hans-Werner Henn [Hen07] §3.4 and then begin the construction of the topological duality resolution. The details of the algebraic duality resolution can be found in [Bea15a].

The two resolutions have complementary features. While we will not try to make this thought completely precise, the duality tower resolution reflects, in an essential way, the fact that the group \mathbb{S}_2^1 is a virtual Poincaré duality group of dimension 3.

The centralizer resolution on the other hand, is much closer to being an Adams-Novikov tower as there is an underlying relative homological algebra in the spirit of Miller [Mil81]. See Remark 3.26 below.

3.1. The centralizer resolution. Henn's centralizer resolutions grew out of his paper [Hen98] which used the centralizers of elementary abelian subgroups of S_n to detect elements in the cohomology of S_n . At the prime 2, this approach needs a slight modification, as the maximal finite 2-group in S_2 is Q_8 , which is not elementary abelian.

Remark 3.1. In (1.9) we defined $G_{24} \subseteq \mathbb{S}_2^1$ as the image of a group of automorphisms of a supersingular elliptic curve. The group \mathbb{S}_2^1 fits into a short exact sequence

$$1 \longrightarrow \mathbb{S}_2^1 \longrightarrow \mathbb{S}_2 \xrightarrow{N} \mathbb{Z}_2 \longrightarrow 1$$

where N is the reduced determinant map of (1.6). Let

$$\pi = 1 + 2\omega$$

be an element of \mathbb{S}_2 , where $\omega \in \mathbb{W}^\times$ is a cube root of unity. Notice that π is not an element of \mathbb{S}_2^1 because $N(\pi) = 3$. Then we define $G'_{24} := \pi G_{24} \pi^{-1} \subseteq \mathbb{S}_2^1$. This is a subgroup isomorphic to G_{24} , but not conjugate to G_{24} in \mathbb{S}_2^1 .

Note that multiplication by π defines an equivalence $E^{hG_{24}} \simeq E^{hG'_{24}}$. For complete details on this and more, see [Bea15a].

We now have the following result from §3.4 of [Hen07]. This is the *algebraic centralizer resolution*.

Theorem 3.2. *There is an exact sequence of continuous \mathbb{S}_2^1 -modules*

$$(3.3) \quad \begin{aligned} 0 \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/C_2]] \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/C_4]] \\ \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_2 \rightarrow 0. \end{aligned}$$

The map ϵ is the sum of the augmentation maps.

We will call this a resolution, even though the terms are not projective as $\mathbb{Z}_2[[\mathbb{S}_2^1]]$ -modules. It is an \mathcal{F} -projective resolution, an idea we explore below in Remark 3.26.

Remark 3.4. Suppose we write

$$\begin{aligned} P_0 &= \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \times \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] \\ P_1 &= \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \times \mathbb{Z}_2[[\mathbb{S}_2^1/C_4]] \\ P_2 &= \mathbb{Z}_2[[\mathbb{S}_2^1/C_2]] \\ P_3 &= \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]]. \end{aligned}$$

Then for any profinite \mathbb{S}_2^1 -module M , we get a spectral sequence

$$E_1^{p,q} \cong \text{Ext}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}^q(P_p, M) \implies H^{p+q}(\mathbb{S}_2^1, M).$$

The E_1 -terms can all be written as group cohomology groups; for example

$$E_1^{0,q} \cong H^q(G_{24}, M) \times H^q(G'_{24}, M).$$

We will call this the *algebraic centralizer resolution spectral sequence*. If M is a \mathbb{G}_2 -module, such as $E_n X$ for some spectrum X , then multiplication by π induces an isomorphism $H^q(G_{24}, M) \cong H^q(G'_{24}, M)$.

Remark 3.5. We can induce the resolution (3.3) of \mathbb{S}_2^1 -modules up to a resolution of \mathbb{G}_2 -modules and obtain an exact sequence

$$\begin{aligned} 0 \rightarrow \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \rightarrow \mathbb{Z}_3[[\mathbb{G}_2/C_2]] \rightarrow \mathbb{Z}_2[[\mathbb{G}_2/C_6]] \oplus \mathbb{Z}_2[[\mathbb{G}_2/C_4]] \\ \rightarrow \mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{G}_2/G'_{24}]] \rightarrow \mathbb{Z}_2[[\mathbb{G}_2/\mathbb{S}_2^1]] \rightarrow 0 . \end{aligned}$$

Since G'_{24} is conjugate to G_{24} in \mathbb{G}_2 , we have $\mathbb{Z}_2[[\mathbb{G}_2/G_{24}]] \cong \mathbb{Z}_2[[\mathbb{G}_2/G'_{24}]]$ as \mathbb{G}_2 -modules and we have made that substitution. If F is any closed subgroup of \mathbb{G}_2 , then the equivalence of (1.16) gives us an isomorphism of twisted \mathbb{G}_2 -modules

$$\mathrm{Hom}_{\mathbb{Z}_2[[\mathbb{G}_2]]}(\mathbb{Z}_2[[\mathbb{G}_2/F]], E_*) \cong E_* E^{hF}.$$

Combining these observations, we get an exact sequence of twisted \mathbb{G}_2 -modules

$$(3.6) \quad 0 \rightarrow E_* E^{h\mathbb{S}_2^1} \rightarrow \begin{array}{c} E_* E^{hG_{24}} \\ \times \\ E_* E^{hG_{24}} \end{array} \rightarrow \begin{array}{c} E_* E^{hC_6} \\ \times \\ E_* E^{hC_4} \end{array} \rightarrow E_* E^{hC_2} \rightarrow E_* E^{hC_6} \rightarrow 0 .$$

We then have the following result; this is the *topological centralizer resolution* of Theorem 12 of [Hen07].

Theorem 3.7. *The algebraic resolution of 3.6 can be realized by a sequence of spectra*

$$E^{h\mathbb{S}_2^1} \xrightarrow{p} \begin{array}{c} E^{hG_{24}} \\ \times \\ E^{hG_{24}} \end{array} \rightarrow \begin{array}{c} E^{hC_6} \\ \times \\ E^{hC_4} \end{array} \rightarrow E^{hC_2} \rightarrow E^{hC_6}$$

All compositions and all Toda brackets are zero modulo indeterminacy.

Remark 3.8. The vanishing of the Toda brackets in this result has several implications. To explain these and for future reference we write, echoing the notation of Remark 3.4:

$$(3.9) \quad \begin{aligned} F_0 &= E^{hG_{24}} \times E^{hG_{24}} \\ F_1 &= E^{hC_6} \times E^{hC_4} \\ F_2 &= E^{hC_2} \\ F_3 &= E^{hC_6} . \end{aligned}$$

Then the resolution of Theorem 3.7 can be refined to a tower of fibrations under $E^{h\mathbb{S}_2^1}$:

$$(3.10) \quad \begin{array}{ccccccc} E^{h\mathbb{S}_2^1} & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & E^{hG_{24}} \times E^{hG_{24}} = F_0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{-3} F_3 & & \Sigma^{-2} F_2 & & \Sigma^{-1} F_1 & & \end{array}$$

Alternatively we could refine the resolution into a tower over $E^{h\mathbb{S}_2^1}$. Let us write

$$\begin{array}{ccc} X & \xleftarrow{\quad} & Z \\ & \searrow \quad \nearrow & \\ & Y & \end{array}$$

for a cofiber sequence (i.e., a triangle) $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$. Then we have a diagram of cofiber sequences

$$(3.11) \quad \begin{array}{ccccccc} E^{h\mathbb{S}_2^1} & \xleftarrow{\quad} & C_1 & \xleftarrow{\quad} & C_2 & \xleftarrow{\quad} & C_3 \xrightarrow{\simeq} \\ & \searrow p & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & F_0 & & F_1 & & F_2 & & F_3 \end{array}$$

where each of the compositions $F_{i-1} \rightarrow C_i \rightarrow F_i$ is the map $F_{i-1} \rightarrow F_i$ in the resolution.

The towers (3.10) and (3.11) determine each other. This is because there is a diagram with rows and columns cofibration sequences

$$(3.12) \quad \begin{array}{ccccc} \Sigma^{-(s+1)}C_{s+1} & \longrightarrow & E^{h\mathbb{S}_2^1} & \longrightarrow & Y_s \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-s}C_s & \longrightarrow & E^{h\mathbb{S}_2^1} & \longrightarrow & Y_{s-1} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma^{-s}F_s & \longrightarrow & * & \longrightarrow & \Sigma^{-s+1}F_s \end{array}$$

Using the tower over $E^{h\mathbb{S}_2^1}$ of (3.11) we get a number of spectral sequences; for example, if Y is any spectrum, we have get a spectral sequence for the function spectrum $F(Y, E^{h\mathbb{S}_2^1})$

$$(3.13) \quad E_1^{s,t} = \pi_t F(Y, F_s) \implies \pi_{t-s} F(Y, E^{h\mathbb{S}_2^1}).$$

Up to isomorphism, this spectral sequence can be obtained from the tower of (3.10); this follows from (3.12).

Remark 3.14. It is direct to calculate E_*C_s and E_*Y_s for the layers of the two towers. If we define $K_s \subseteq E_*F_s$ to be the image of $E_*F_{s-1} \rightarrow E_*F_s$, then $E_*C_s \cong K_s$ and, more, if we apply E_* to (3.11) we get a collection of short exact sequences:

$$\begin{array}{ccccccc} E_*E^{h\mathbb{S}_2^1} & \xleftarrow{\quad 0 \quad} & E_*C_1 & \xleftarrow{\quad 0 \quad} & E_*C_2 & \xleftarrow{\quad 0 \quad} & E_*C_3 \xrightarrow{\simeq} \\ & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\ & E_*F_0 & & E_*E_*F_1 & & E_*F_2 & & E_*F_3 . \end{array}$$

Notice that this implies that each of the dotted arrows of (3.11) has Adams-Novikov filtration one. Finally, the cofibration sequence $E^{h\mathbb{S}_2^1} \rightarrow Y_s \rightarrow \Sigma^{-s}C_{s+1}$ induces a short exact sequence

$$0 \rightarrow E_*E^{h\mathbb{S}_2^1} \rightarrow E_*Y_s \rightarrow \Sigma^{-s}K_{s+1} \rightarrow 0$$

which is split if $s > 0$.

3.2. The duality resolution, first steps. We have the *algebraic duality resolution* from [Bea15a]. The groups G_{24} and G'_{24} are defined in Remark 3.1.

Theorem 3.15. *There is an exact sequence of continuous \mathbb{S}_2^1 -modules*

$$(3.16) \quad 0 \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] \rightarrow \mathbb{Z}_3[[\mathbb{S}_2^1/C_6]] \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \rightarrow \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \xrightarrow{\epsilon} \mathbb{Z}_2 \rightarrow 0$$

where ϵ is the augmentation.

Remark 3.17. As in Remark 3.4 we get a spectral sequence. Suppose we write

$$\begin{aligned} Q_0 &= \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \\ Q_1 &= Q_2 = \mathbb{Z}_2[[\mathbb{S}_2^1/C_6]] \\ Q_3 &= \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] \end{aligned}$$

Then for any profinite \mathbb{S}_2^1 -module M , such as $E_k X = (E_2)_k X$ for some spectrum X , we get a spectral sequence

$$E_1^{p,q} \cong \text{Ext}_{\mathbb{Z}_2[[\mathbb{S}_2^1]]}^q(Q_p, M) \implies H^{p+q}(\mathbb{S}_2^1, M),$$

which we will call the *algebraic duality resolution spectral sequence*.

As in Remark 3.5 and (3.6) we immediately have the following consequence.

Corollary 3.18. *There is an exact sequence of twisted \mathbb{G}_2 -modules*

$$0 \rightarrow E_* E^{h\mathbb{S}_2^1} \rightarrow E_* E^{hG_{24}} \rightarrow E_* E^{hC_6} \rightarrow E_* E^{hC_6} \rightarrow E_* E^{hG_{24}} \rightarrow 0.$$

The first of these maps is induced by the inclusion of fixed points.

We'd now like to prove the following result, paralleling Theorem 3.7; it also appears in [Hen07]. The main work of the next two sections and, indeed, the main theorem of this paper is to identify X .

Proposition 3.19. *The algebraic resolution of 3.18 can be realized by a sequence of spectra*

$$E^{h\mathbb{S}_2^1} \xrightarrow{q} E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow E^{hC_6} \rightarrow X$$

with $E_* X \cong E_* E^{hG_{24}}$ as a twisted \mathbb{G}_2 -module. All compositions and all Toda brackets are zero modulo indeterminacy.

Remark 3.20. A consequence of the last sentence of this result is that this resolution can be refined to a tower of fibrations under $E^{h\mathbb{S}_2^1}$

$$(3.21) \quad \begin{array}{ccccccc} E^{h\mathbb{S}_2^1} & \longrightarrow & Z_2 & \longrightarrow & Z_1 & \longrightarrow & E^{hG_{24}} \\ \uparrow & & \uparrow & & \uparrow & & \\ \Sigma^{-3} X & & \Sigma^{-2} E^{hC_6} & & \Sigma^{-1} E^{hC_6} & & \end{array}$$

or to a tower over $E^{h\mathbb{S}_2^1}$

$$(3.22) \quad \begin{array}{ccccccc} E^{h\mathbb{S}_2^1} & \xleftarrow{\quad} & D_1 & \xleftarrow{\quad} & D_2 & \xleftarrow{\quad} & D_3 \\ \searrow q & & \nearrow & & \nearrow & & \nearrow \\ & E^{hG_{24}} & & E^{hC_6} & & E^{hC_6} & \\ & & & & & & \nearrow \simeq \\ & & & & & & X \end{array}$$

As in Remark 3.14 the dotted arrows have Adams-Novikov filtration 1. Examining this last diagram, we see that X can be *defined* as the cofiber of $D_2 \rightarrow E^{hC_6}$ and

it will follow, as in Remark 3.14, that $E_*X \cong E_*E^{hG_{24}}$. Thus Proposition 3.19 is equivalent to the following result. See also [Hen07] or [Bob14].

Lemma 3.23. *The truncated resolution*

$$0 \rightarrow E_*E^{h\mathbb{S}_2^1} \rightarrow E_*E^{hG_{24}} \rightarrow E_*E^{hC_6} \rightarrow E_*E^{hC_6}$$

can be realized by a sequence of spectra

$$E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow E^{hC_6}$$

such that all compositions are zero and the one Toda bracket is zero modulo indeterminacy.

Proof. The map $E^{h\mathbb{S}_2^1} \xrightarrow{q} E^{hG_{24}}$ is simply the inclusion of fixed points. To realize the other maps and to show the compositions are zero, we prove that

$$\pi_0 F(E^{hF}, E^{hC_6}) \longrightarrow \text{Hom}_{E_0[[\mathbb{G}_2]]}(E_0E^{hF}, E_0E^{hC_6})$$

is an isomorphism for $F = \mathbb{S}_2^1$ or $F = G_{24}$. To do this, first combine the results of (1.21) with the computations of homotopy groups of fixed point spectra of §2.1 to see that

$$\pi_0 F(E^{hF}, E^{hC_6}) = \pi_0 E[[\mathbb{G}_2/F]]^{hC_6} \cong H^0(C_6, E_0[[\mathbb{G}_2/F]]).$$

Then we can finish the argument by using (1.18) to show

$$\begin{aligned} \text{Hom}_{E_0[[\mathbb{G}_2]]}(E_0E^{hF}, E_0E^{hC_6}) &\cong \text{Hom}_{E^0[[\mathbb{G}_2]]}(E^0[[\mathbb{G}_2/C_6]], E^0[[\mathbb{G}_2/F]]) \\ &\cong H^0(C_6, E^0[[\mathbb{G}_2/F]]) . \end{aligned}$$

This leaves the Toda bracket. To see that it is zero modulo indeterminacy we show that the indeterminacy is the entire group. To be specific, we show that the inclusion of the $E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}}$ induces a surjection

$$\pi_* F(E^{hG_{24}}, E^{hC_6}) \longrightarrow \pi_* F(E^{h\mathbb{S}_2^1}, E^{hC_6}).$$

We can rewrite this as the map induced by the inclusion $G_{24} \rightarrow \mathbb{S}_2^1$:

$$\pi_* E[[\mathbb{G}_2/G_{24}]]^{hC_6} \longrightarrow \pi_* E[[\mathbb{G}_2/\mathbb{S}_2^1]]^{hC_6}.$$

Using (1.22) we then rewrite this as a map

$$(3.24) \quad \lim_i \prod_{C_6 \setminus \mathbb{G}_n / U_i} \pi_* E^{hF_x} \longrightarrow \lim_i \prod_{C_6 \setminus \mathbb{G}_n / V_i} \pi_* E^{hG_x}$$

where $U_i \subseteq V_i$ are open subgroups, $\cap U_i = G_{24}$, $\cap V_i \cong \mathbb{S}_2^1$, and for any double coset $C_6 x U_i$

$$G_x = C_6 \cap x V_i x^{-1} \subseteq F_x = C_6 \cap x U_i x^{-1}.$$

We have that $C_2 \subseteq G_x \subseteq F_x \subseteq C_6$, since $C_2 \subseteq \mathbb{G}_2$ is in the center. To show that this map of (3.24) is onto, we will show that each of the maps

$$(3.25) \quad \prod_{C_6 \setminus \mathbb{G}_n / U_i} \pi_* E^{hF_x} \longrightarrow \prod_{C_6 \setminus \mathbb{G}_n / V_i} \pi_* E^{hG_x}$$

is onto. Since the products are finite, the results of Section 2.1 shows that the kernel of this map is compact; therefore, the $\lim - \lim^1$ short exact sequence will show that the map of (3.24) is onto.

To finish the argument, we use (1.20) to see that the map of (3.25) is given by the quotient map on double cosets and the transfer maps on the factors:

$$\mathrm{tr}_x : E^{hF_x} \longrightarrow E^{hG_x}.$$

These maps are the transfers associated to the inclusion $G_x \subseteq F_x$. Since the only possibilities here are that $G_x = F_x$ or $G_x = C_2$ and $F_x = C_6$, the map tr_x is surjective on homotopy. \square

3.3. Comparing the two resolutions. There is a map from the centralizer tower to the duality tower; we will not prove that here. In the end we will only need a small part of the data given by such a map, and what we need is in Remark 3.29.

Remark 3.26. The underlying algebra for the centralizer resolutions fits well the relative homological algebra usually deployed in building an Adams-Novikov tower; this goes back to Miller in [Mil81], among other sources.

Here is more detail. Let \mathcal{F} be the set of conjugacy classes of finite subgroups of \mathbb{S}_2^1 . A continuous \mathbb{S}_2^1 -module P is \mathcal{F} -projective if the natural map

$$\bigoplus_{F \in \mathcal{F}} \mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[[F]]} P \longrightarrow P$$

is split surjective, where F runs over representatives for the classes in \mathcal{F} . The class of \mathcal{F} -projectives is the smallest class of continuous \mathbb{S}_2^1 -modules closed under sums, retracts, and containing all induced modules $\mathbb{Z}_2[[\mathbb{S}_2^1]] \otimes_{\mathbb{Z}_2[[F]]} M$, where M is a continuous F -module.

The class of \mathcal{F} -projectives defines a class of \mathcal{F} -exact morphisms, there are enough \mathcal{F} -projectives, there are \mathcal{F} -projective resolutions, and so on. All of this and more is discussed in §3.5 of [Hen07].

Comparing the two towers now begins with the following result, see Remark (d) after Proposition 17 in [Hen07].

Proposition 3.27. *The centralizer resolution (3.3) is an \mathcal{F} -projective resolution of the trivial \mathbb{S}_2^1 -module \mathbb{Z}_2 .*

Thus if we write $P_\bullet \rightarrow \mathbb{Z}_2$ for the centralizer resolution (3.3) and $Q_\bullet \rightarrow \mathbb{Z}_2$ for the duality resolution (3.16), then standard homological algebra gives us a map of resolutions, unique up to chain homotopy

$$\begin{array}{ccc} Q_\bullet & \longrightarrow & \mathbb{Z}_2 \\ f_\bullet \downarrow & & \downarrow = \\ P_\bullet & \longrightarrow & \mathbb{Z}_2 \end{array}.$$

We won't need to be explicit about this map, but we add that the map $f_0 : Q_0 \rightarrow P_0$ can be chosen to be the inclusion onto the first factor

$$(3.28) \quad Q_0 = \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \xrightarrow{i_1} \mathbb{Z}_2[[\mathbb{S}_2^1/G_{24}]] \oplus \mathbb{Z}_2[[\mathbb{S}_2^1/G'_{24}]] = P_0.$$

Remark 3.29. This immediately gives a map from the centralizer resolution spectral sequence of Remark 3.4 to the duality resolution spectral sequence of Remark 3.17. This map is independent of the choice of f_\bullet at the E_2 -page. This can be lifted to a map from the centralizer tower to the duality tower, although we don't need that here and won't prove it. We note that (3.28) implies there is a commutative diagram where the horizontal maps are the edge homomorphisms of the two spectral sequences

$$\begin{array}{ccc} H^*(\mathbb{S}_2^1, E_*) & \xrightarrow{p^*} & H^*(\mathbb{G}_{24}, E_*) \times H^*(\mathbb{G}_{24}, E_*) \\ \downarrow = & & \downarrow (f_0)^* \\ H^*(\mathbb{S}_2^1, E_*) & \xrightarrow{q_*} & H^*(\mathbb{G}_{24}, E_*) \end{array}$$

and $(f_0)^*$ is projection onto the first factor. This can be realized by a diagram of spectra, where the map f_0 is again projection onto the first factor

$$\begin{array}{ccc} E^{h\mathbb{S}_2^1} & \xrightarrow{p} & E^{hG_{24}} \times E^{hG_{24}} \\ \downarrow = & & \downarrow f_0 \\ E^{h\mathbb{S}_2^1} & \xrightarrow{q} & E^{hG_{24}} \end{array}.$$

4. CONSTRUCTING ELEMENTS IN $\pi_{192k+48}X$

We now turn to the analysis of the homotopy groups of X , where $\Sigma^{-3}X$ is the top fiber in the duality tower; see Proposition 3.19. We have an isomorphism of Morava modules $E_*X \cong E_*E^{hG_{24}}$ and hence a spectral sequence

$$H^*(G_{24}, E_*) \Longrightarrow \pi_*X.$$

The E_2 -term is discussed in Theorem 2.15. In this section we show, roughly, that $\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$ is a permanent cycle—which would certainly be necessary if our main result is true. The exact result is below in Corollary 4.7. In the next section, we will use this and a mapping space argument to finish the identification of the homotopy type X .

The statements and the arguments in this section have a rather fussy nature because the spectrum X has no *a priori* ring or module structure and, in particular, the \mathbb{W} -algebra structure on $H^*(G_{24}, E_*)$ does not immediately extend to a \mathbb{W} -module structure on π_*X .

The results of this section were among the main results in the first author's thesis [Bob14] and the key ideas for the entire project can be found there.

We begin by combining Remark 1.30 and Lemma 2.20 to obtain the following result. Note that $45 \equiv 5$ modulo 8, so there is no contributions from the *bo*-patterns in that degree. In all degrees the *bo*-patterns lie in Adams-Novikov filtration at most 2.

Lemma 4.1. *There is an isomorphism*

$$\mathbb{F}_4 \cong \pi_{45}E^{hG_{24}}.$$

We can chose an \mathbb{F}_4 generator detected by the class

$$\Delta \bar{\kappa} \eta \in H^5(G_{24}, E_{50}).$$

The class $\Delta \bar{\kappa}^2 \eta^2 \in H^{10}(G_{24}, E_{76})$ is a non-zero permanent cycle detecting an \mathbb{F}_4 generator of the subgroup of $\pi_{66} E^{hG_{24}}$ consisting of the elements of Adams-Novikov filtration greater than 2.

Let $p : E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}} \times E^{hG_{24}}$ be the augmentation in the topological centralizer resolution of Theorem 3.7. This is the same map as from the top to the bottom of the centralizer resolution tower (3.10).

Lemma 4.2. *Let $k \in \mathbb{Z}$. The map*

$$p_* : \pi_{192k+45} E^{h\mathbb{S}_2^1} \longrightarrow \pi_{192k+45} (E^{hG_{24}} \times E^{hG_{24}})$$

is surjective. If $x \in \pi_{192k+45} E^{h\mathbb{S}_2^1}$ has the property that $p_(x) \neq 0$, then x has Adams-Novikov filtration at most 5, $x\bar{\kappa}\eta \neq 0$, and $x\bar{\kappa}\eta$ is detected by a class of Adams-Novikov filtration at most 10.*

Proof. For the first statement we examine the homotopy spectral sequence of the centralizer tower (3.13). In this case this spectral sequence reads

$$\pi_t F_s \implies \pi_{t-s} E^{h\mathbb{S}_2^1}.$$

The fibers F_s are spelled out in (3.9). We are asking that the edge homomorphism

$$p_* : \pi_{192k+45} E^{h\mathbb{S}_2^1} \longrightarrow \pi_{192k+45} F_0$$

be surjective. The crucial input is that

$$\pi_k E^{hC_6} \subseteq \pi_k E^{hC_2} = 0$$

for $k = 45, 46$, and 47 and that $\pi_{45} E^{hC_4} = 0$. See Proposition 2.9 and Figure 2.

The final statement follows from Lemma 4.1. \square

Now let $q : E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}}$ be the augmentation in the topological duality resolution of Theorem 3.19. This is also the projection from the top to the bottom of the duality resolution tower (3.21). Let $i : \Sigma^{-3} X \rightarrow E^{h\mathbb{S}_2^1}$ be the map from the top fiber of the duality tower. Consider the commutative diagram

$$(4.3) \quad \begin{array}{ccccc} \pi_{192k+45} \Sigma^{-3} X & \xrightarrow{i_*} & \pi_{192k+45} E^{h\mathbb{S}_2^1} & \xrightarrow{=} & \pi_{192k+45} E^{h\mathbb{S}_2^1} \\ \downarrow r & & \downarrow p_* & & \downarrow q_* \\ \mathbb{F}_4 & \longrightarrow & \pi_{192k+45} (E^{hG_{24}} \times E^{hG_{24}}) & \xrightarrow{(f_0)_*} & \pi_{192k+45} E^{hG_{24}}. \end{array}$$

The bottom row is short exact and induced by the map between the resolutions. See Remark 3.29. The map r is defined by this diagram and the fact that the composition

$$\pi_* \Sigma^{-3} X \longrightarrow \pi_* E^{h\mathbb{S}_2^1} \xrightarrow{q_*} \pi_* E^{hG_{24}}$$

is zero.

Proposition 4.4. *The map*

$$r : \pi_{192k+45}\Sigma^{-3}X \rightarrow \mathbb{F}_4$$

is surjective. If $y \in \pi_{192k+45}\Sigma^{-3}X$ is any class so that $r(y) \neq 0$, then y is detected by a class

$$f \stackrel{\text{def}}{=} f(j)\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48}) \cong W[[j]]\Delta^{8k+2}.$$

Furthermore

$$f\bar{\kappa}\eta \in H^5(G_{24}, E_{192k+73})$$

*is a non-zero permanent cycle in the spectral sequence for π_*X .*

Proof. In the diagram (4.3), the map p_* is onto, and any element x in the kernel of $(f_0)_*$ must have filtration at least 1 in the homotopy spectral sequence of the duality tower. Since $\pi_k E^{hC_6} = 0$ for $k = 46$ and $k = 47$, by Proposition 2.12, any such element must be the image of class $y \in \pi_*\Sigma^{-3}X$. This shows r is surjective.

The map $\Sigma^{-3}X \rightarrow E^{hS_2^1}$ raises Adams-Novikov filtration by 3; see the diagram of (3.22) and the remarks thereafter. If $r(y) = p_*i_*(y) \neq 0$, then by Lemma 4.2, y must have Adams-Novikov filtration at most 2; however, by Theorem 2.15 and the chart of Figure 3 we have that

$$H^1(G_{24}, E_{192k+49}) = 0 = H^2(G_{24}, E_{192k+50}).$$

Thus y must have filtration 0. Similarly $0 \neq y\bar{\kappa}\eta$ must have filtration at least five and at most 7. Again we examine the chart Figure 3 to find it must have filtration 5 and be detected in group cohomology, as claimed. \square

Recall that we are writing $S/2$ for the mod 2 Moore spectrum.

Proposition 4.5. *Let $y \in \pi_{192k+48}X$ be detected by*

$$f = f(j)\Delta^{8k+2} \in H^0(G_{24}, E_*).$$

If $r(y) \neq 0$, then f and $f\bar{\kappa}\eta$ are non-zero permanent cycles in the spectral sequence

$$H^*(G_{24}, E_*/2) \Longrightarrow \pi_*(X \wedge S/2).$$

Proof. This follows from Proposition 4.4 and the fact that

$$H^5(G_{24}, E_{192k+73}) \rightarrow H^5(G_{24}, E_{192k+73}/2)$$

is injective. This last statement can be deduced from the long exact sequence in cohomology induced by the short exact sequence $0 \rightarrow E_* \rightarrow E_* \rightarrow E_*/2 \rightarrow 0$ and the fact that

$$H^6(G_{24}, E_{192k+73}) = 0.$$

See Figure 3. \square

The crucial theorem then becomes:

Theorem 4.6. *Let $y \in \pi_{192k+48}X$ and let*

$$f = f(j)\Delta^{8k+2} \in H^0(G_{24}, E_*)$$

be the image of y under the edge homomorphism

$$\pi_*X \longrightarrow H^0(G_{24}, E_*).$$

If $f(j) \equiv 0$ modulo $(2, j)$, then $r(y) = 0$.

Proof. We will show that if $f(0) \equiv 0$ modulo 2, then $f\bar{\kappa} = 0$ in $E_4^{*,*}(X \wedge S/2)$. We can then apply Proposition 4.5.

Under the assumption $f(0) \equiv 0$ modulo 2 we have

$$f = jg(j)\Delta^{8k+2} \in H^0(G_{24}, E_*/2).$$

We will show that in the Adams-Novikov Spectral Sequence we have

$$d_3(v_1^8 g(j)\Delta^{8k+2}\mu) = f\bar{\kappa}.$$

The result will follow.

We appeal to Theorem 2.15, Remark 2.16, Theorem 2.15, and the chart of Figure 3. We have

$$j\bar{\kappa} = c_4^2 \eta^4$$

and, hence, that

$$f\bar{\kappa} = jg(j)\Delta^{8k+2}\bar{\kappa} = c_4^2 g(j)\Delta^{8k+2}\eta^4.$$

Since $f\bar{\kappa}$ is a d_3 -cycle, d_3 is η -linear, and

$$\eta^4 : E_3^{192k+48,0}/2 \cong H^0(G_{24}, (E/2)_{192k+48}) \rightarrow H^4(G_{24}, E_{192k+52}) \cong E_3^{192k+48+4,8}$$

is injective we have that

$$d_3(c_4^2 g(j)\Delta^{8k+2}) = 0.$$

It now follows from Lemma 2.21 that

$$d_3(c_4^2 g(j)\Delta^{8k+2}\mu) = c_4^2 g(j)\Delta^{8k+2}\eta^4 = f\bar{\kappa}.$$

This is what we promised. \square

The next result has a slightly complicated statement because we don't know yet that π_*X is a \mathbb{W} -module.

Corollary 4.7. *There is a commutative diagram*

$$\begin{array}{ccc} \pi_{192k+48}X & \longrightarrow & H^0(G_{24}, E_{192k+48}) \\ r \downarrow & & \downarrow \epsilon \\ \mathbb{F}_4 & \xrightarrow{\cong} & \mathbb{F}_4 \end{array}$$

where the bottom map is some possibly non-trivial isomorphism of groups and

$$\epsilon(f(j)\Delta^{8k+2}) = f(0) \quad \text{mod } (2, j).$$

There are homotopy classes $x_{k,i} \in \pi_{192k+48}X$, $i = 1, 2$ detected by classes

$$f_i(j)\Delta^{8k+2} \in H^0(G_{24}, E_{192k+48})$$

so that $f_1(0)$ and $f_2(0)$ span \mathbb{F}_4 as an \mathbb{F}_2 vector space.

Proof. This is an immediate consequence of Theorem 4.6. \square

5. THE MAPPING SPACE ARGUMENT

We would like to extend the results of the Section 4 in the following way. Let $\iota : S^0 \rightarrow E^{hG_{48}}$ be the unit and r the map defined in (4.3). Note that we are using $E^{hG_{48}}$ here.

Theorem 5.1. *The composite*

$$\pi_{48}F(E^{hG_{48}}, X) \xrightarrow{\iota^*} \pi_{48}X \xrightarrow{r} \mathbb{F}_4$$

is surjective.

We can use this result to build maps out of $E^{hG_{48}}$ as follows. Consider the following diagram. Note we are using that $E^{-48} = E_{48}$.

$$(5.2) \quad \begin{array}{ccc} \pi_{48}(E^{hG_{48}}, X) & \xrightarrow{i^*} & \pi_{48}X \\ \downarrow H & & \downarrow H \\ \mathrm{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}[[\mathbb{G}_2/G_{48}]]) & \xrightarrow{\iota^*} & \mathrm{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}) \\ & & \downarrow \cong \\ & & \mathbb{W}[[j]]\Delta^2 \cong (E_{48})^{G_{24}} \\ & & \downarrow \epsilon \\ & & \mathbb{F}_4 \end{array}$$

where the maps labelled H are the Hurewicz maps for $E^*(-)$ and the map ϵ reduces mod $(2, j)$. By Corollary 4.7, the vertical composition on the right is r up to some automorphism of \mathbb{F}_4 . Proposition 4.4 and Corollary 4.7 then yield the following corollary to Theorem 5.1.

Corollary 5.3. *Let $f(j)\Delta^2 \in H^0(G_{24}, E_{48})$. Then there is a map*

$$\phi : \Sigma^{48}E^{hG_{48}} \rightarrow X$$

so that $\iota_(\phi) \equiv f(0)$ modulo 2.*

We will use this result to show that there is an equivalence $\Sigma^{48}E^{hG_{24}} \rightarrow X$. See Theorem 5.8 below.

We now begin the proof of Theorem 5.1. Let $p : E^{hS_2^1} \rightarrow E^{hG_{24}} \times E^{hG_{24}}$ be the projection from the top to the bottom of the centralizer resolution tower.

Lemma 5.4. *Let $k \in \mathbb{Z}$. The map*

$$p_* : \pi_{192k+45}F(E^{hG_{48}}, E^{hS_2^1}) \rightarrow \pi_{192k+45}(F(E^{hG_{48}}, E^{hG_{24}}) \times F(E^{hG_{48}}, E^{hG_{24}}))$$

is surjective.

Proof. We apply $F(E^{hG_{48}}, -)$ to the centralizer tower and examine the resulting spectral sequence in homotopy. See (3.13). The spectral sequence reads

$$\pi_t F(E^{hG_{48}}, F_s) \implies \pi_{t-s} F(E^{hG_{48}}, E^{h\mathbb{S}_2^1})$$

and the fibers F_s are described in (3.9). Thus we need to know

$$\begin{aligned} 0 &= \pi_{192k+45} F(E^{hG_{48}}, E^{hC_6} \vee E^{hC_4}) \\ &= \pi_{192k+46} F(E^{hG_{48}}, E^{hC_2}) \\ &= \pi_{192k+47} F(E^{hG_{48}}, E^{hC_6}) . \end{aligned}$$

We can use (1.21). Note that C_2 is central, so all of the subgroups F_x contain C_2 . Therefore, the crucial input is as before:

$$\pi_k E^{hC_6} \subseteq \pi_k E^{hC_2} = 0$$

for $k = 45, 46$, and 47 and that $\pi_{45} E^{hC_4} = 0$. See Propositions 2.8, 2.9, and 2.12. See also Figure 2. \square

Let $q : E^{h\mathbb{S}_2^1} \rightarrow E^{hG_{24}}$ be the projection from the top to the bottom of the duality resolution tower. Using Remark 3.29 we now can produce a commutative diagram, where we have abbreviated $F(E^{hG_{48}}, Y)$ as $F(Y)$ and we're writing $n = 192k + 45$.

$$(5.5) \quad \begin{array}{ccccc} \pi_n F(\Sigma^{-3} X) & \longrightarrow & \pi_n F(E^{h\mathbb{S}_2^1}) & \xrightarrow{=} & \pi_n F(E^{h\mathbb{S}_2^1}) \\ \downarrow r & & \downarrow p_* & & \downarrow q_* \\ \pi_n F(E^{hG_{24}}) & \longrightarrow & \pi_n (F(E^{hG_{24}}) \times F(E^{hG_{24}})) & \xrightarrow{j_*} & \pi_n F(E^{hG_{24}}) \\ \downarrow \iota_* & & \downarrow \iota_* & & \downarrow \iota_* \\ \pi_n E^{hG_{24}} & \longrightarrow & \pi_n (E^{hG_{24}} \times E^{hG_{24}}) & \xrightarrow{i_*} & \pi_n E^{hG_{24}} \end{array}$$

The lower two rows are split short exact, the maps labelled ι_* are all onto, and the maps p_* and q_* are onto. Theorem 5.1 follows from Lemma 1.23 and the following.

Lemma 5.6. *The map*

$$r : \pi_{192k+45} F(E^{hG_{48}}, \Sigma^{-3} X) \rightarrow \pi_{192k+45} F(E^{hG_{48}}, E^{hG_{24}})$$

is onto.

Proof. The proof is an exact copy of the first part of the argument for Proposition 4.4, generalized to mappings spaces; that is, we examine the homotopy spectral sequence built from the duality tower for $F(E^{hG_{48}}, E^{h\mathbb{S}_2^1})$. In the diagram (5.5), the map q_* is onto, and any element x in the kernel of j_* must have filtration at least 1 in the homotopy spectral sequence of the duality tower. Since $\pi_k F(E^{hG_{48}}, E^{hC_6}) = 0$ for $k = 46$ and $k = 47$, using (1.21) and Propositions 2.8 and 2.9, any such element must be the image of class $\pi_* F(E^{hG_{48}}, \Sigma^{-3} X)$. \square

Remark 5.7. Note that all of these arguments would work with replacing $E^{hG_{48}}$ with $E^{hG_{24}}$.

The next result is our main theorem.

Theorem 5.8. *There is an equivalence*

$$\Sigma^{48} E^{hG_{24}} \xrightarrow{\simeq} X$$

realizing the given isomorphism of Morava modules

$$E_* E^{hG_{24}} \xrightarrow{\cong} E_* X.$$

Proof. We actually use the given isomorphism of Morava modules to produce a (non-equivariant) equivalence

$$C_2^+ \wedge \Sigma^{48} E^{hG_{48}} \simeq \Sigma^{48} (E^{hG_{48}} \vee E^{hG_{48}}) \rightarrow X.$$

Here $C_2 = \text{Gal} = \text{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Then we will apply Lemma 1.28.

We begin with some algebra. Recall from Remark 1.12 that the 2-Sylow subgroup $S_2 \subseteq \mathbb{S}_2$ can be decomposed as $K \rtimes Q_8$. Since $G_{24} = Q_8 \rtimes \mathbb{F}_4^\times$ and $G_{48} = G_{24} \rtimes \text{Gal}$ we have that $E^*[[\mathbb{G}_2/G_{24}]]$ and $E^*[[\mathbb{G}_2/G_{48}]]$ are free $E^*[[K]]$ modules of rank 2 and rank 1 respectively. Since K is finitely generated pro-2-group, the ring $E^0[[K]]$ is a complete local ring with maximal ideal \mathfrak{m}_K given by the kernel of the reduced augmentation

$$E^0[[K]] \longrightarrow E^0 \longrightarrow \mathbb{F}_4.$$

We will produce a map

$$f : \Sigma^{48} (E^{hG_{48}} \vee E^{hG_{48}}) \rightarrow X$$

so that the map of $E^0[[K]]$ -modules

$$E^* f : E^{48} X \longrightarrow E^{48} (\Sigma^{48} (E^{hG_{48}} \vee E^{hG_{48}}))$$

is an isomorphism modulo \mathfrak{m}_K . Then, by the appropriate variant of Nakayama's Lemma (see Lemma 4.3 of [GHMR05]), it will be an isomorphism of $E^0[[K]]$ -modules and, hence, of $E^0[[\mathbb{G}_2]]$ -modules, as required.

Since $G_{24} \cong Q_8 \rtimes \mathbb{F}_4^\times$ and $\mathbb{G}_2 \cong ((K \rtimes Q_8) \rtimes \mathbb{F}_4^\times) \rtimes \text{Gal}$, we have an isomorphism of K -sets $\mathbb{G}_2/G_{24} \cong K \sqcup K\phi$ where ϕ is the Frobenius in the Galois group. Then we have the following commutative diagram. It is an expansion of the diagram (5.2).

$$\begin{array}{ccc} \text{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}[[\mathbb{G}_2/G_{48}]]) & \longrightarrow & \text{Hom}_K(E^0[[K]] \oplus E^0[[K\phi]], E^{-48}[[K]]) \\ \downarrow \iota^* & & \downarrow \iota^* \\ \text{Hom}_{\mathbb{G}_2}(E^0[[\mathbb{G}_2/G_{24}]], E^{-48}) & \longrightarrow & \text{Hom}_K(E^0[[K]] \oplus E^0[[K\phi]], E^{-48}) \\ \downarrow \cong & & \downarrow \subseteq \\ (E_{48})^{G_{24}} \cong \mathbb{W}[[j]] & \longrightarrow & E_{48}^2 \cong (\mathbb{W}[[u_1]])^2 \\ \downarrow & & \downarrow \\ \mathbb{F}_4 & \longrightarrow & \mathbb{F}_4^2. \end{array}$$

The top two horizontal maps are forgetful maps, remembering only the K action. The third horizontal map is the composition

$$\mathbb{W}[[j]] \xrightarrow{\subseteq} \mathbb{W}[[u_1]] \xrightarrow{1 \times \phi} (\mathbb{W}[[u_1]])^2.$$

and the bottom map is $x \mapsto (x, \phi(x))$. The top vertical maps are induced by the map $E^*(E^{hG_{48}}) \rightarrow E^*S^0$ given by the unit, the middle vertical map evaluates a homomorphism at $1 \in E^0[[\mathbb{G}_2/G_{24}]]$; note we are again using that $E^{-48} \cong E_{48}$. The final map is reduction modulo the maximal ideals in both cases.

By Corollary 5.3 we can produce two maps

$$f_i : \Sigma^{48} E^{hG_{48}} \rightarrow X, \quad i = 1, 2$$

so that $(f_i)_*(\iota) \equiv \omega^i \Delta^2$ modulo $(2, j)$, where $\omega \in \mathbb{F}_4$ is the primitive cube root of unity. We now examine the fate of

$$E^0(f_i) : E^0[[\mathbb{G}_2/G_{24}]] \longrightarrow E^{-48}[[\mathbb{G}_2/G_{48}]]$$

as we work from the upper left to the bottom right of this diagram. Using the formulas of the previous paragraph we have

$$E^0(f_1) \mapsto (\omega, \omega^2) \quad \text{and} \quad E^0(f_2) \mapsto (\omega^2, \omega).$$

Finally, let

$$f = f_1 \vee f_2 : \Sigma^{48}(E^{hG_{48}} \vee E^{hG_{48}}) \rightarrow X.$$

If we apply $E^*(-)$ to this map we get a map

$$E^*f : E^0[[\mathbb{G}_2/G_{24}]] \rightarrow E^{-48}[[\mathbb{G}_2/G_{48}]] \times E^{-48}[[\mathbb{G}_2/G_{48}]]$$

which yields a map of K -modules

$$E^*f : E^0[[K]]^2 \longrightarrow E^{-48}[[K]]^2$$

which, modulo the maximal ideal \mathfrak{m}_K , gives the map

$$\mathbb{F}_4^2 \longrightarrow \mathbb{F}_4^2$$

given by the matrix

$$\begin{pmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{pmatrix}$$

with determinant $\omega^2 + \omega = 1$. Thus E^*f is an isomorphism, as needed. \square

We can now complete the proof of Theorem 0.1 and construct the topological duality resolution. Proposition 3.19 and Theorem 5.8 imply the following. Recall from Theorem 2.9 that E^{hC_6} is 48-periodic.

Corollary 5.9. *There exists a resolution of $E^{hS_2^1}$ in the $K(2)$ -local category at the prime 2*

$$E^{hS_2^1} \rightarrow E^{hG_{24}} \rightarrow E^{hC_6} \rightarrow \Sigma^{48} E^{hC_6} \rightarrow \Sigma^{48} E^{hG_{24}}.$$

Remark 5.10. In [GHMR05], working at the prime 3, the authors were able to produce a topological resolution for $L_{K(2)}S^0 = E^{hG_2}$ itself by the same methods that produced the resolution for $E^{hG_2^1}$. The resolution for the sphere was essentially a double of that for $E^{hG_2^1}$. Not only do the methods of [GHMR05] not apply to the case $p = 2$, it's very unlikely that there is a topological resolution of the sphere,

or for $E^{h\mathbb{S}_2}$, which could be obtained by doubling the resolution of $E^{h\mathbb{S}_2^1}$. There are any number of difficulties, but the first obstacle is that we have only a semi-direct decomposition $\mathbb{S}_2^1 \rtimes \mathbb{Z}_2 \cong \mathbb{S}_2$ at $p = 2$, rather than the product decomposition $\mathbb{G}_2^1 \times \mathbb{Z}_3 \cong \mathbb{G}_2$ at the prime 3. This makes the algebra much harder, and it only gets worse from there. Other short topological resolutions are possible, of course, and could be very instructive. This is the subject of current research by Agnès Beaudry and Hans-Werner Henn.

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